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Original Article

New stability and boundedness results to Volterra integro-differential equations with delay

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Abstract In this paper, we consider a certain non-linear Volterra integro-differential equations with delay. We study stability and boundedness of solutions. The technique of proof involves defining suitable Lyapunov functionals. Our results improve and extend the results obtained in literature.

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1. Introduction

In the last years, the qualitative properties of Volterra integro-differential equations without delay have been discussed by many researches. In particular, the reader can referee to the papers of Becker [1], Burton [2,3], Burton and Mahfoud [4,5] Diamandescu [6], Hara et al. [7], Miller [8], Staffans [9], Tunc [10], Vanualailai and Nakagiri [11] and the books of Burton [12], Corduneanu [13], Gripenberg et al. [14] and the references cited therein for some works done on qualitative properties of various Volterra integro-differential equations without delay. An important tool to discuss the qualitative properties of solutions of ordinary and functional differential equations and integro-differential equations is the Lyapunov's direct method. Theoretically this method is very appealing, and there are numerous applications where it is natural to use it. The key requirement of the method is to find a positive definite function or functional which is non-increasing along solutions.

However, it is a quite difficult task to find a suitable Lyapunov function or functional for a non-linear ordinary or functional differential equation and a non-linear functional Volterra integro- differential equation. The situation becomes more difficult when we replace an ordinary or a functional differential equation with a functional integro-differential equation. By this time, the construction of Lyapunov functions and functionals for non-linear differential and integro-differential systems remains as an open problem in the literature. Besides, in the literature, there are a few papers on the qualitative behaviors of Volterra integro-differential equations with delay. See, for ex-

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ample, the recent papers of Adıvar and Raffoul [15], Graef and Tunc [16], Raffoul [17] and Raffoul and Unal [18].

In 2003, Vanualailai and Nakagiri [11] considered the nonlinear Volterra integro-differential equation without delay,

$$\frac{d}{dt}[x(t)] = A(t)f(x(t)) + \int_0^t B(t,s)g(x(s))ds, \tag{1}$$

where $t \geq 0$, $x \in \Re$, $A(t) : [0,\infty) \to (-\infty, 0)$, $f, g : \Re \to \Re$ are continuous functions, and $B(t,s)$ is a continuous function for $0 \leq s \leq t < \infty$. Vanualailai and Nakagiri [11] studied the stability of solutions of Eq. (1) by defining a suitable Lyapunov functional.

In this paper, we consider the nonlinear the Volterra integro-differential equation with delay

$$x'(t) = -a(t)f(x(t)) + \int_{t-\tau}^t B(t,s)g(x(s))ds + p(t), \tag{2}$$

where $t \geq 0$, τ is a positive constant, fixed delay, $x \in \Re$, $a(t) : [0,\infty) \to (0,\infty)$, $p : [0,\infty) \to \Re$, $f, g : \Re \to \Re$ are continuous functions with $f(0) = g(0) = 0$, and $B(t,s)$ is a continuous function for $0 \leq s \leq t < \infty$.

We investigate the stability of zero solution and boundedness of all solutions of Eq. (2) by defining new suitable Lyapunov functionals, when $p(t) \equiv 0$ and $p(t) \neq 0$, respectively.

It follows that Vanualailai and Nakagiri [11] considered a Volterra integro-differential equation without delay. However, in this paper, we consider a Volterra integero-differential equation with delay. Besides, Vanualailai and Nakagiri [11] discussed the stability of the zero solution of Eq. (1). However, beside the stability of zero solution, we also discuss the boundedness of solutions of Eq. (2), when $p(t) \equiv 0$ and $p(t) \neq 0$, respectively. Further, Eq. (2) includes and extends the equations discussed by Vanualailai and Nakagiri [11], when $\tau = 0$.

We give some basic information related Eq. (2).

We use the following notation throughout this paper.

For any $t_0 \geq 0$ and initial function $\varphi \in [t_0 - \tau, t_0]$, let $x(t) = x(t, t_0, \varphi)$ denote the solution of Eq. (2) on $[t_0 - \tau, \infty)$ such that $x(t) = \varphi(t)$ on $\varphi \in [t_0 - \tau, t_0]$.

Let $C[t_0, t_1]$ and $C[t_0, \infty)$ denote the set of all continuous real-valued functions on $[t_0, t_1]$ and $[t_0, \infty)$, respectively.

For $\varphi \in C[0, t_0]$, $|\varphi|_{t_0} := \sup\{|\varphi(t)| : 0 \leq t \leq t_0\}$.

Definition. The zero solution of Eq. (2) is stable if for each $\varepsilon > 0$ and each $t_0 \geq 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $\varphi \in C[0, t_0]$ with $|\varphi(t)|_{t_0} < \delta$ implies that $|x(t, t_0, \varphi)| < \varepsilon$ for all $t \geq t_0$.

Let $p(t) = 0$ in Eq. (1).

The following theorem is need for the stability result of this theorem.

Theorem 1 (Driver [19]). *If there exists a functional $V(t, \varphi(.))$, defined whenever $t \geq t_0 \geq 0$ and $\varphi \in C([0,t], \Re)$, such that*

(i) $V(t, 0) \equiv 0$, *V is continuous in t and locally Lipschitz in φ,*
(ii) $V(t, \varphi(.)) \geq W(|\varphi(t)|)$, $W : [0,\infty) \to [0,\infty)$ *is a continuous function with $W(0) = 0$, $W(r) > 0$ if $r > 0$, and W strictly increasing (positive definiteness), and*
(iii) $V'(t, \varphi(.)) \leq 0$,

then the zero solution of Eq. (2) is stable, and

$$V(t, \varphi(.)) = V(t, \varphi(s) : 0 \leq s \leq t)$$

is called a Lyapunov functional for Eq. (2).

2. The main results

We state some assumptions on the functions that are appearing in Eq. (2).

A. Assumptions

(A1) There exist positive constants α, m, J, M and N such that
$f(0) = 0$, $g(0) = 0$, $g^2(x) \leq m^2 f^2(x)$ if $|x| \leq M$,
$\alpha > 4$ such that $4x^2 \leq (\alpha - 4)f^2(x)$ if $|x| \leq N$.

(A2) $a(t) > 0$ for $t \geq 0$, $B(t,s)$ is continuous for $0 \leq s \leq t < \infty$,
$J \geq 1$, $\frac{1}{4a(t)} \int_{t-\tau}^t |B(t,s)|ds < \frac{1}{J}$ for every $t \geq s - \tau \geq 0$,
$\int_{t-\tau}^\infty |B(u+\tau, s)|du$ is defined and continuous for $0 \leq s - \tau \leq t < \infty$.
$a(t) - k\int_{t-\tau}^\infty |B(u+\tau, t)|du \geq 0$ for every $t \geq s - \tau \geq 0$.

For the case $p(t) = 0$ in Eq. (2), we have the following result.

Theorem 2. *Assume conditions (A1) and (A2) hold. If $k = \frac{m^2(1+\alpha)}{J}$, then the zero solution of Eq. (2) is stable.*

Proof. We introduce a functional $V_0 = V_0(t) = V_0(t, x(t))$ defined by

$$V_0 = \frac{1}{2}x^2 + \sqrt{\alpha}\int_0^x \sqrt{f(u)u}\,du + \frac{1}{2}\alpha\int_0^x f(u)du + k\int_0^t \int_{t-\tau}^\infty |B(u+\tau, s)|du f^2(x(s))ds, \tag{3}$$

where k is a positive constant to be determined later in the proof. □

It is clear that the functional V_0 is positive definite.

Differentiating the functional V_0 with respect to t, we obtain from (3) that

$$V_0' = xx' + \sqrt{\alpha}\sqrt{f(x)x}x' + \frac{1}{2}\alpha f(x)x' + k\int_{t-\tau}^\infty |B(u+\tau, t)|du f^2(x) - k\int_0^t |B(t,s)| f^2(x(s))ds. \tag{4}$$

Then, it is clear that

$$\begin{aligned}
xx' &= -a(t)xf(x) + x\int_{t-\tau}^t B(t,s)g(x(s))ds \\
&= -a(t)xf(x) - \left[\sqrt{a(t)}x - \frac{1}{2\sqrt{a(t)}}\int_{t-\tau}^t B(t,s)g(x(s))ds\right]^2 \\
&\quad + a(t)x^2 + \frac{1}{4a(t)}\left[\int_{t-\tau}^t B(t,s)g(x(s))ds\right]^2 \\
&\leq -a(t)xf(x) + a(t)x^2 + \frac{1}{4a(t)}\left[\int_{t-\tau}^t B(t,s)g(x(s))ds\right]^2 \\
&\leq -a(t)xf(x) + a(t)x^2 \\
&\quad + \frac{1}{4a(t)}\int_{t-\tau}^t |B(t,s)|ds \int_{t-\tau}^t |B(t,s)|g^2(x(s))ds \\
&\leq -a(t)xf(x) + a(t)x^2
\end{aligned}$$

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