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Original Article

On certain subclasses of analytic and bi-univalent functions

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Abstract In this paper, we introduced two interesting subclasses of the function class σ of analytic and bi-univalent functions in the open unit disk U. Estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging to these classes are determined. Certain special cases are also indicated.

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1. Introduction

Let A denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, $\mathbb{C}$ being, as usual, the set of complex numbers. We also denote by S the subclass of all functions in A which are univalent in U. Let S_s^* be the subclass of S consisting of functions of the form (1.1) satisfying

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0, \quad z \in U. \tag{1.2}$$

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi [1] (see also Robertson [2], Stankiewicz [3], Wu [4] and Owa et al. [5]). Das and Singh [6] introduced another class C_s namely, convex functions with respect to symmetric points and satisfying the condition

$$\mathrm{Re}\left(\frac{(zf'(z))'}{(f(z) - f(-z))'}\right) > 0, \quad z \in U. \tag{1.3}$$

If f and g are analytic functions in U, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function φ, which (by definition) is analytic in U with $\varphi(0) = 0$ and $|\varphi(z)| < 1$ for all $z \in U$, such that $f(z) = g(\varphi(z))$, $z \in U$.

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Furthermore, if the function g is univalent in U, then we have the following equivalence

$$f(z) \prec g(z)(z \in U) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

For each $f \in S$, the Koebe one-quarter theorem [7] ensures the image of U under f contains a disk of radius 1/4. Thus every univalent function $f \in S$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

$$f(f^{-1}(\omega)) = \omega \quad (|\omega|) < r_0(f); \quad r_0(f) \geq \frac{1}{4}\Big).$$

In fact, the inverse function $g = f^{-1}$ is given by

$$g(\omega) = f^{-1}(\omega) = w - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^2 - 5a_2a_3 + a_4)\omega^4 + \cdots .$$

A function $f \in A$ is said to be bi-univalent in U if f and f^{-1} are univalent in U. Let σ denote the class of bi-univalent functions in U given by (1.1). The familiar Koebe function is not a member of σ because it maps the unit disk U univalently onto the entire complex plane minus a slit along the line $\frac{-1}{4}$ to $-\infty$. Hence the image domain does not contain the unit disk U.

In 1985 Branges [8] proved the celebrated Bieberbach Conjecture which states that, for each $f(z) \in S$ given by the Taylor–Maclaurin series expansion (1.1), the following coefficient inequality holds true:

$$|a_n| \leq n \quad (n \in N - \{1\}),$$

N being the set of positive integers. The class of analytic bi-univalent functions was first introduced and studied by Lewin [9], where it was proved that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [10] improved Lewin's result to $|a_2| \leq \sqrt{2}$. Brannan and Taha [11] and Taha [12] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike and convex functions. They introduced bi-starlike functions and bi-convex functions and found non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. For further historical account of functions in the class σ, see the work by Srivastava et al. [13] (see also [11,14]). In fact, the above-cited recent pioneering work of Srivastava et al. [13] has apparently revived the study of analytic and bi-univalent functions in recent years; it was followed by such works as those by Frasin and Aouf [15], Xu et al. [16,17], Hayami and Owa [18], and others (see, for example, [19–35]).

In the present paper, certain subclasses of the bi-univalent function class σ were introduced, and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ were found.

2. Coefficient estimates

In the sequel, it is assumed that ϕ is an analytic function with positive real part in the unit disk U, satisfying $\phi(0) = 1, \phi'(0) > 0$, and $\phi(U)$ is symmetric with respect to the real axis. Such a function has a Taylor series of the form

$$\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots \ (B_1 > 0). \tag{2.1}$$

Suppose that $u(z)$ and $v(z)$ are analytic in the unit disk U with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(z)| < 1$, and suppose that

$$u(z) = b_1z + \sum_{n=2}^{\infty} b_nz^n, \quad v(z) = c_1z + \sum_{n=2}^{\infty} c_nz^n \quad (z \in U). \tag{2.2}$$

It is well known that (see Nehari [36, p. 172])

$$|b_1| \leq 1, \quad |b_2| \leq 1 - |b_1|^2, \quad |c_1| \leq 1, \quad |c_2| \leq 1 - |c_1|^2. \tag{2.3}$$

By a simple calculation, we have

$$\phi(u(z)) = 1 + B_1b_1z + (B_1b_2 + B_2b_1^2)z^2 + \cdots \ (z \in U), \tag{2.4}$$

and

$$\phi(v(\omega)) = 1 + B_1c_1\omega + (B_1c_2 + B_2c_1^2)\omega^2 + \cdots \ (\omega \in U). \tag{2.5}$$

Definition 1. A function $f \in \sigma$ given by (1.1) is said to be in the class $S_\sigma(\alpha, \phi)(0 \leq \alpha \leq 1)$ if the following conditions are satisfied:

$$(1-\alpha)\frac{2zf'(z)}{f(z) - f(-z)} + \alpha\frac{2(zf'(z))'}{(f(z) - f(-z))'} \prec \phi(z) \ (z \in U)$$

and

$$(1-\alpha)\frac{2\omega g'(\omega)}{g(\omega) - g(-\omega)} + \alpha\frac{2(\omega g'(\omega))'}{(g(\omega) - g(-\omega))'} \prec \phi(\omega), \ (\omega \in U),$$

where $g(\omega) := f^{-1}(\omega)$.

Theorem 1. *If $f(z)$ given by (1.1) be in the class $S_\sigma(\alpha, \phi)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{4(1+\alpha)^2B_1 + 2\left|(1+2\alpha)B_1^2 - 2(1+\alpha)^2B_2\right|}} \tag{2.6}$$

and

$$|a_3| \leq \begin{cases} \frac{B_1}{2(1+2\alpha)} & if\ B_1 \leq \frac{2(1+\alpha)^2}{1+2\alpha}, \\ \left(1 - \frac{2(1+\alpha)^2}{(1+2\alpha)B_1}\right) & if\ B_1 > \frac{2(1+\alpha)^2}{1+2\alpha} \\ \frac{B_1^3}{4(1+\alpha)^2B_1 + 2\left|(1+2\alpha)B_1^2 - 2(1+\alpha)^2B_2\right|} + \frac{B_1}{2(1+2\alpha)}. & \end{cases} \tag{2.7}$$

Proof. Let $f \in S_\sigma(\alpha, \phi)$. Then there are analytic functions u, v: $U \to U$ given by (2.2) such that

$$(1-\alpha)\frac{2zf'(z)}{f(z) - f(-z)} + \alpha\frac{2(zf'(z))'}{(f(z) - f(-z))'} = \phi(u(z)) \tag{2.8}$$

and

$$(1-\alpha)\frac{2\omega g'(\omega)}{g(\omega) - g(-\omega)} + \alpha\frac{2(\omega g'(\omega))'}{(g(\omega) - g(-\omega))'} = \phi(v(\omega)). \tag{2.9}$$

Since