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Original Article

Properties of certain subclass of *p*-valent meromorphic functions associated with certain linear operator

Rabha M. El-Ashwah [a], **Alaa H. Hassan** [b,*]

[a] *Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt*
[b] *Department of Mathematics, Faculty of Science, Zagazig University, Zagazig 44519, Egypt*

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Abstract We investigate several inclusion relationships of certain subclass of *p*-valent meromorphic functions defined in the punctured unit disc, having a pole of order *p* at the origin. The subclass under investigation is defined by using certain linear operator defined by combining two integral operators.

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1. Introduction

Let Σ_p denotes the subclass of meromorphic functions of the form

$$f(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are analytic in the punctured unit disc $U^* = U\backslash\{0\}$, where $U = \{z \in \mathbb{C} : |z| < 1\}$.

For two functions $f(z)$ and $g(z)$, analytic in U, we say that $f(z)$ is subordinate to $g(z)$ in U, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $\omega(z)$ which (by definition) is analytic in U, satisfying the following conditions (see [1,2]):

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1; \qquad (z \in U)$$

such that

$$f(z) = g(\omega(z)); \quad (z \in U),$$

Indeed it is known that

$$f(z) \prec g(z) \quad (z \in U) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

* Corresponding author. Tel.: +20 1066757776.
E-mail addresses: r_elashwah@yahoo.com (Rabha M. El-Ashwah), alaahassan1986@yahoo.com (Alaa H. Hassan).
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In particular, if the function $g(z)$ is univalent in U, we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in U) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Following the recent work of El-Ashwah [3], for a function $f(z) \in \Sigma_p$, given by (1.1), also, for $\lambda,\ \ell > 0$ and $m \in \mathbb{N}_0(\mathbb{N}_0 = \mathbb{N} \cup \{0\})$, the integral operator $L_p^m(\lambda, \ell) : \Sigma_p \longrightarrow \Sigma_p$ is defined as follows:

$$L_p^m(\lambda, \ell)f(z) = \begin{cases} f(z); & (m = 0), \\ \frac{\ell}{\lambda} z^{-p-\frac{\ell}{\lambda}} \int_0^z t^{\left(\frac{\ell}{\lambda}+p-1\right)} L_p^{m-1}(\lambda, \ell) f(t) dt; & (m = 1, 2, \ldots). \end{cases} \tag{1.2}$$

Also, following the recent work of El-Ashwah and Hassan [4], for a function $f(z) \in \Sigma_p$, given by (1.1), also, for $\mu > 0$, $a, c \in \mathbb{C}$ and $Re(c - a) \geq 0$, the integral operator $J_{p,\mu}^{a,c} : \Sigma_p \longrightarrow \Sigma_p$ is defined as follows:

$$J_{p,\mu}^{a,c} f(z) = \begin{cases} f(z); & (a = c), \\ \frac{\Gamma(c - p\mu)}{\Gamma(a - p\mu)\Gamma(c - a)} \int_0^1 t^{a-1}(1 - t)^{c-a-1} f(zt^\mu) dt; & (Re(c - a) > 0). \end{cases} \tag{1.3}$$

By iterations of the integral operators $L_p^m(\lambda, \ell)$ defined by (1.2) and $J_{p,\mu}^{a,c}$ defined by (1.3), we define the linear operator

$$I_{\lambda,\ell}^{p,m}(a, c, \mu) : \Sigma_p \longrightarrow \Sigma_p \tag{1.4}$$

for the purpose of this paper by:

$$I_{\lambda,\ell}^{p,m}(a, c, \mu) f(z) := L_p^m(\lambda, \ell)\left(J_{p,\mu}^{a,c} f(z)\right) = J_{p,\mu}^{a,c}\left(L_p^m(\lambda, \ell) f(z)\right). \tag{1.5}$$

Now, it is easily to see that the operator $I_{\lambda,\ell}^{p,m}(a, c, \mu)$ can be expressed as follows:

$$I_{\lambda,\ell}^{p,m}(a, c, \mu) f(z) = z^{-p} + \frac{\Gamma(c - p\mu)}{\Gamma(a - p\mu)} \times \sum_{k=1-p}^{\infty} \frac{\Gamma(a + \mu k)}{\Gamma(c + \mu k)} \left[\frac{\ell}{\ell + \lambda(k + p)}\right]^m a_k z^k,$$

$$(\mu > 0;\ a, c \in \mathbb{C},\ Re(a) > p\mu,\ Re(c - a) \geq 0;\ \ell > 0;\ \lambda > 0;\ m \in \mathbb{N}_0;\ p \in \mathbb{N}). \tag{1.6}$$

In view of (1.2)–(1.5), it is clear that

$$I_{\lambda,\ell}^{p,0}(a, c, \mu) f(z) = J_{p,\mu}^{a,c} f(z) \quad \text{and} \quad I_{\lambda,\ell}^{p,m}(a, a, \mu) f(z) = L_p^m(\lambda, \ell) f(z). \tag{1.7}$$

Using (1.6), we can obtain the following recurrence relations of the operator $I_{\lambda,\ell}^{p,m}(a, c, \mu)$, which are necessary for our investigations

$$z\left(I_{\lambda,\ell}^{p,m}(a, c, \mu) f(z)\right)' = \frac{a - p\mu}{\mu} I_{\lambda,\ell}^{p,m}(a + 1, c, \mu) f(z) - \frac{a}{\mu} I_{\lambda,\ell}^{p,m}(a, c, \mu) f(z). \tag{1.8}$$

and

$$z\left(I_{\lambda,\ell}^{p,m}(a, c + 1, \mu) f(z)\right)' = \frac{c - p\mu}{\mu} I_{\lambda,\ell}^{p,m}(a, c, \mu) f(z) - \frac{c}{\mu} I_{\lambda,\ell}^{p,m}(a, c + 1, \mu) f(z). \tag{1.9}$$

Also

$$z\left(I_{\lambda,\ell}^{p,m+1}(a, c, \mu) f(z)\right)' = \frac{\ell}{\lambda} I_{\lambda,\ell}^{p,m}(a, c, \mu) f(z) - \frac{\ell + \lambda p}{\lambda} I_{\lambda,\ell}^{p,m+1}(a, c, \mu) f(z). \tag{1.10}$$

The operator $I_{\lambda,\ell}^{p,m}(a, c, \mu)$ defined by (1.7) has been extensively studied by many authors with suitable restrictions on the parameters as follows:

(i) $I_{\lambda,\ell}^{1,-n}(a, c, \mu) = I_{\lambda,\ell}^{n}(a, c, \mu) f(z) (\mu>0;\ a, c\in\mathbb{C},\ Re(c-a) \geq 0,\ Re(a)>\mu;\ \ell>0;\ \lambda>0;\ n\in\mathbb{Z})$ (see El-Ashwah [5]);
(ii) $I_{\lambda,\ell}^{p,m}(p + \nu, p + 1, 1) = I_{p,\nu}^{m}(\lambda, \ell) f(z) (m \in \mathbb{N}_0;\ \lambda, \ell, \nu>0;\ p \in \mathbb{N})$ (see El-Ashwah and Aouf [6]);
(iii) $I_{\nu,\lambda}^{1,m}(a + 1, c + 1, 1) f(z) = \Im_{\lambda,\nu}^{m}(a, c) f(z) \quad (\lambda, \nu>0;\ a \in \mathbb{C};\ c \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0)$ (see Raina and Sharma [7]);
(iv) $I_{\lambda,\ell}^{p,0}(a + p, c + p, 1) f(z) = \ell_p(a, c) f(z) \quad (a \in \mathbb{R};\ c \in \mathbb{R}\backslash \mathbb{Z}_0^-,\ \mathbb{Z}_0^- = \{0, 1, 2, \ldots\};\ p \in \mathbb{N})$ (see Liu and Srivastava [8]);
(v) $I_{1,\lambda}^{1,\beta}(\nu + 1, 2, 1) f(z) = I_{\lambda,\nu}^{\beta} f(z)\ (\beta \geq 0;\ \lambda > 0;\ \nu > 0)$ (see Piejko and Sokół [9]);
(vi) $I_{1,\lambda}^{1,n}(\nu + 1, 2, 1) f(z) = I_{\lambda,\nu}^{n} f(z) \quad (n \in \mathbb{N}_0;\ \lambda > 0;\ \nu > 0)$ (see Cho et al. [10]);
(vii) $I_{\lambda,\ell}^{1,0}(\nu + 1, n + 2, 1) f(z) = \ell_{n,\nu} f(z)\ (n > -1;\ \nu > 0)$ (see Yuan et al. [11]);
(viii) $I_{\lambda,\ell}^{p,0}(n + 2p, p + 1, 1) f(z) = D^{n+p-1} f(z)$ (n is an integer, $n > -p,\ p \in \mathbb{N}$) (see Uralegaddi and Somanatha [12]);
(ix) $I_{1,1}^{p,\alpha}(a, a, \mu) f(z) = P_p^\alpha f(z) (\alpha \geq 0;\ p \in \mathbb{N})$ (see Aqlan et al. [13]);
(x) $I_{1,\beta}^{1,\alpha}(a, a, \mu) f(z) = P_\beta^\alpha f(z) (\alpha, \beta > 0;\ p \in \mathbb{N})$ (see Lashin [14]).

Now, by the help of the linear operator $I_{\lambda,\ell}^{p,m}(a, c, \mu)$, we introduce the subclass $M_{\lambda,\ell}^{p,m}(a, c, \mu; \alpha; A, B)$ of meromorphic functions as follows:

Definition 1. For fixed parameters A, $B(-1 \leq B < A \leq 1)$ and $0 \leq \alpha < p$, the function $f(z) \in \Sigma_p$ is said to be in the class $M_{\lambda,\ell}^{p,m}(a, c, \mu; \alpha; A, B)$ if it satisfies the following subordination condition:

$$\frac{1}{p - \alpha}\left(\frac{-z\left(I_{\lambda,\ell}^{p,m}(a, c, \mu) f(z)\right)'}{I_{\lambda,\ell}^{p,m}(a, c, \mu) f(z)} - \alpha\right) \prec \frac{1 + Az}{1 + Bz} (z \in U), \tag{1.11}$$

$(\mu>0;\ a, c\in\mathbb{C},\ Re(a)>p\mu,\ Re(c-a) \geq 0;\ \ell>0;\ \lambda>0;\ m\in\mathbb{N}_0;\ p\in\mathbb{N})$.

Or, equivalently

$$M_{\lambda,\ell}^{p,m}(a, c, \mu; \alpha; A, B) = \left\{ f(z)\in\Sigma_p : \left| \frac{\frac{z\left(I_{\lambda,\ell}^{p,m}(a,c,\mu) f(z)\right)'}{I_{\lambda,\ell}^{p,m}(a,c,\mu) f(z)} + p}{B\frac{z\left(I_{\lambda,\ell}^{p,m}(a,c,\mu) f(z)\right)'}{I_{\lambda,\ell}^{p,m}(a,c,\mu) f(z)} + [pB + (A-B)(p-\alpha)]} \right| < 1 \right\}. \tag{1.12}$$