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Original Article

ϕ-statistically quasi Cauchy sequences☆

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Abstract Let P denote the space whose elements are finite sets of distinct positive integers. Given any element σ of P, we denote by $p(\sigma)$ the sequence $\{p_n(\sigma)\}$ such that $p_n(\sigma) = 1$ for $n \in \sigma$ and $p_n(\sigma) = 0$ otherwise. Further $P_s = \{\sigma \in P : \sum_{n=1}^{\infty} p_n(\sigma) \le s\}$, i.e. P_s is the set of those σ whose support has cardinality at most s. Let (ϕ_n) be a non-decreasing sequence of positive integers such that $n\phi_{n+1} \le (n+1)\phi_n$ for all $n \in \mathbb{N}$ and the class of all sequences (ϕ_n) is denoted by Φ. Let $E \subseteq \mathbb{N}$. The number $\delta_\phi(E) = \lim_{s\to\infty} \frac{1}{\phi_s}|\{k \in \sigma, \sigma \in P_s : k \in E\}|$ is said to be the ϕ-density of E. A sequence (x_n) of points in $\mathbb{R}$ is ϕ-statistically convergent (or S_ϕ-convergent) to a real number ℓ for every $\varepsilon > 0$ if the set $\{n \in \mathbb{N} : |x_n - \ell| \ge \varepsilon\}$ has ϕ-density zero. We introduce ϕ-statistically ward continuity of a real function. A real function is ϕ-statistically ward continuous if it preserves ϕ-statistically quasi Cauchy sequences where a sequence (x_n) is called to be ϕ-statistically quasi Cauchy (or S_ϕ-quasi Cauchy) when $(\Delta x_n) = (x_{n+1} - x_n)$ is ϕ-statistically convergent to 0. i.e. a sequence (x_n) of points in $\mathbb{R}$ is called ϕ-statistically quasi Cauchy (or S_ϕ-quasi Cauchy) for every $\varepsilon > 0$ if $\{n \in \mathbb{N} : |x_{n+1} - x_n| \ge \varepsilon\}$ has ϕ-density zero. Also we introduce the concept of ϕ-statistically ward compactness and obtain results related to ϕ-statistically ward continuity, ϕ-statistically ward compactness, statistically ward continuity, ward continuity, ward compactness, ordinary compactness, uniform continuity, ordinary continuity, δ-ward continuity, and slowly oscillating continuity.

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1. Introduction

First of all, some definitions and notation will be given in the following. Throughout this paper, $\mathbb{N}$, and $\mathbb{R}$ will denote the set of all positive integers, and the set of all real numbers, respectively. We will use boldface letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, ... for sequences $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, $\mathbf{z} = (z_n)$, ... of terms in $\mathbb{R}$.

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Let us start with basic definitions from the literature. Let $K \subseteq \mathbb{N}$, the set of all natural numbers and $K_n = \{k \leq n : k \in K\}$. Then the *natural density* of K is defined by $\delta(K) = \lim_n n^{-1}|K_n|$ if the limit exists, where the vertical bars indicate the number of elements in the enclosed set.

Fast [1] presented the following definition of statistical convergence for sequences of real numbers. The sequence $x = (x_n)$ is said to be *statistically convergent* to L if for every $\epsilon > 0$, the set $K_\epsilon := \{n \in \mathbb{N} : |x_n - L| \geq \epsilon\}$ has natural density zero, i.e. for each $\epsilon > 0$,

$$\lim_n \frac{1}{n}|\{j \leq n : |x_j - L| \geq \epsilon\}| = 0.$$

In this case, we write S-$\lim x = L$ or $x_n \to L(S)$ and S denotes the set of all statistically convergent sequences. Note that every convergent sequence is statistically convergent but not conversely.

Some basic properties related to the concept of statistical convergence were studied in [2,3]. In 1985, Fridy [4] presented the notion of statistically Cauchy sequence and determined that it is equivalent to statistical convergence. Caserta et al. [5] studied statistical convergence in function sapces while Caserta and Kočinac [6] investigated statistical exhaustivness. For more details on statistical convergence we refer to [7–12].

A sequence (x_n) of points in $\mathbb{R}$ is called quasi-Cauchy if (Δx_n) is a null sequence where $\Delta x_n = x_{n+1} - x_n$. In [13] Burton and Coleman named these sequences as “quasi-Cauchy” and in [14] Çakallı used the term “ward convergent to 0” sequences. From now on in this paper we also prefer to using the term “quasi-Cauchy” to using the term “ward convergent to 0” for simplicity. In terms of quasi-Cauchy we restate the definitions of ward compactness and ward continuity as follows: a function f is ward continuous if it preserves quasi-Cauchy sequences, i.e. $(f(x_n))$ is quasi-Cauchy whenever (x_n) is, and a subset E of $\mathbb{R}$ is ward compact if any sequence $\mathbf{x} = (x_n)$ of points in E has a quasi-Cauchy subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of the sequence $\mathbf{x}$.

It is known that a sequence (x_n) of points in $\mathbb{R}$, the set of real numbers, is slowly oscillating if

$$\lim_{\lambda \to 1^+} \overline{\lim}_n \max_{n+1 \leq k \leq [\lambda n]} |x_k - x_n| = 0$$

where $[\lambda n]$ denotes the integer part of λn. This is equivalent to the following if $(x_m - x_n) \to 0$ whenever $1 \leq \frac{m}{n} \to 1$ as $m, n \to \infty$. Using $\varepsilon > 0$ s and δ s this is also equivalent to the case when for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and $N = N(\varepsilon)$ such that $|x_m - x_n| < \varepsilon$ if $n \geq N(\varepsilon)$ and $n \leq m \leq (1+\delta)n$ (see [15]).

A function f defined on a subset E of $\mathbb{R}$ is called slowly oscillating continuous if it preserves slowly oscillating sequences, i.e. $(f(x_n))$ is slowly oscillating whenever (x_n) is.

Cakalli and Hazarika [16] introduced the concept of ideal quasi Cauchy sequences and proved some results related to ideal ward continuity and ideal ward compactness.

A method G is called regular if every convergent sequence $\mathbf{x} = (x_n)$ is G-convergent with $G(\mathbf{x}) = \lim \mathbf{x}$. A method is called subsequential if whenever $\mathbf{x}$ is G-convergent with $G(\mathbf{x}) = \ell$, then there is a subsequence (x_{n_k}) of $\mathbf{x}$ with $\lim_k x_{n_k} = \ell$. Recently, Cakalli gave new sequential definitions of compactness and slowly oscillating compactness in [15] [17] and [18].

2. Main results

Let P denote the space whose elements are finite sets of distinct positive integers. Given any element σ of P, we denote by $p(\sigma)$ the sequence $\{p_n(\sigma)\}$ such that $p_n(\sigma) = 1$ for $n \in \sigma$ and $p_n(\sigma) = 0$ otherwise. Further $P_s = \{\sigma \in P : \sum_{n=1}^{\infty} p_n(\sigma) \leq s\}$, i.e. P_s is the set of those σ whose support has cardinality at most s. Let (ϕ_n) be a non-decreasing sequence of positive integers such that $n\phi_{n+1} \leq (n+1)\phi_n$ for all $n \in \mathbb{N}$ and the class of all sequences (ϕ_n) is denoted by Φ. For details on class of all sequences (ϕ_n) we refer to [19–21].

Example 2.1. $\phi_n = (2n+1)$ and $\phi_n = (2n+2)$ for all $n \in \mathbb{N}$ are members in Φ, but $\phi_n = (n^2)$ and $\phi_n = (2n-1)$ for all $n \in \mathbb{N}$ are not members in Φ.

Definition 2.1. Let $E \subseteq \mathbb{N}$. The number $\delta_\phi(E) = \lim_{s\to\infty} \frac{1}{\phi_s}|\{k \in \sigma, \sigma \in P_s : k \in E\}|$ is said to be the ϕ-density of E.

Definition 2.2. A sequence (x_n) of points in $\mathbb{R}$ is ϕ-statistically convergent (or S_ϕ-convergent) to a real number ℓ for every $\varepsilon > 0$ if the set $\{n \in \mathbb{N} : |x_n - \ell| \geq \varepsilon\}$ has ϕ-density zero.

Definition 2.3. A sequence $\mathbf{x} = (x_n)$ is S_ϕ-ward convergent to a number ℓ if S_ϕ-$\lim_{n\to\infty} \Delta x_n = \ell$ where $\Delta x_n = x_{n+1} - x_n$. For the special case $\ell = 0$ we say that $\mathbf{x}$ is ϕ-statistically quasi-Cauchy, or S_ϕ-quasi-Cauchy, in place of S_ϕ-ward convergent to 0. Thus a sequence (x_n) of points of $\mathbb{R}$ is S_ϕ-quasi-Cauchy if (Δx_n) is S_ϕ-convergent to 0. We denote ΔS_ϕ the set of all ϕ-statistically quasi Cauchy sequences of points in $\mathbb{R}$.

Example 2.2. Consider a sequence $\mathbf{x} = (x_n) = (\sqrt{n})$. Then it is clear that the sequence (x_n) is ϕ-statistically quasi Cauchy.

Definition 2.4. A subset E of $\mathbb{R}$ is called S_ϕ-sequentially compact if whenever (x_n) is a sequence of points in E there is S_ϕ-convergent subsequence $\mathbf{y} = (y_k) = (x_{n_k})$ of (x_n) such that S_ϕ-$\lim \mathbf{y}$ is in E.

Theorem 2.5. *A subset of $\mathbb{R}$ is sequentially compact if and only if it is S_ϕ-sequentially compact.*

Proof. The proof easily follows from Corollary 3 on page 597 in [17], so is omitted. □

Definition 2.6. A function $f : E \to \mathbb{R}$ is S_ϕ-sequentially continuous at a point x_0 if, given a sequence (x_n) of points in E, S_ϕ-$\lim \mathbf{x} = x_0$ implies that S_ϕ-$\lim f(\mathbf{x}) = f(x_0)$.

Theorem 2.7. *Any S_ϕ-sequentially continuous function at a point x_0 is continuous at x_0 in the ordinary sense.*

Proof. Let f be any S_ϕ-sequentially continuous function at point x_0, Since any proper admissible ideal is a regular subsequential method, it follows from Theorem 13 on page 316 in [22] that f is continuous in the ordinary sense. □

Theorem 2.8. *Any continuous function at a point x_0 is S_ϕ-sequentially continuous at x_0.*

Corollary 2.9. *A function is S_ϕ-sequentially continuous at a point x_0 if and only if it is continuous at x_0.*

Corollary 2.10. *For any regular subsequential method G, a function is G-sequentially continuous at a point x_0, then it is S_ϕ-sequentially continuous at x_0.*

Proof. The proof follows from Theorem 13 on page 316 in [22]. □