Journal of the Egyptian Mathematical Society (2016) 24, 244–249

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

Original Article

A new 2-inner product on the space of *p*-summable sequences

Şükran Konca [a,*], Mochammad Idris [b], Hendra Gunawan [b]

[a] Department of Mathematics, Bitlis Eren University, 13000, Bitlis, Turkey
[b] Department of Mathematics, Institute of Technology Bandung, Bandung 40132, Indonesia

Received 14 February 2015; revised 29 May 2015; accepted 1 July 2015

Keywords

2-inner product spaces;
2-normed spaces;
p-summable sequences;
Weights

Abstract In this paper, we wish to define a 2-inner product, non-standard, possibly with weights, on ℓ^p. For this purpose, we aim to obtain a different 2-norm $\|.,.\|_{2,v,w}$, which is not equivalent to the usual 2-norm $\|.,.\|_p$ on ℓ^p (except with the condition $p = 2$), satisfying the parallelogram law. We discuss the properties of the induced 2-norm $\|.,.\|_{2,v,w}$ and its relationships with the usual 2-norm on ℓ^p. We also find that the 2-inner product $\langle .,.|.\rangle_{v,w}$ is actually defined on a larger space.

2010 MATHEMATICS SUBJECT CLASSIFICATION: Primary 46C50; Secondary 46B20; 46C05; 46C15; 46B99

1. Introduction

By $\ell^p = \ell^p(\mathbb{R})$ we denote the space of all p-summable sequences of real numbers. For $p \neq 2$, $(\ell^p, \|x\|_p)$ is not an inner product space, since the usual norm $\|x\|_p = (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}}$ on ℓ^p does not satisfy the parallelogram law. One alternative is to define a semi-inner product on ℓ^p as in [1], but having a semi-inner product is not as nice as having an inner product.

In [2], Gunawan defined a usual 2-norm $\|.,.\|_p$ on the space of p-summable sequences of real numbers. The usual 2-norm $\|.,.\|_p$ also is not a 2-inner product with $p \neq 2$ because it does not satisfy the parallelogram law.

In this paper, we eventually wish to define a 2-inner product $\langle .,.|.\rangle_{v,w}$, non-standard, possibly with weights, on ℓ^p, so that orthogonality and many other notions on this space can be defined. For this purpose, we aim to obtain a different 2-norm $\|.,.\|_{2,v,w}$, which is not equivalent to the usual 2-norm $\|.,.\|_p$ on ℓ^p (except with the condition $p = 2$), not only satisfies the parallelogram law, but also it is non-standard. For $p > 2$, we also obtain a result which describes how the weighted 2-inner product space is associated to the weights. We discuss the properties of the induced 2 -norm $\|.,.\|_{2,v,w}$ and its relationships with

* Corresponding author. Tel.: +904342285170.
E-mail addresses: skonca@beu.edu.tr, skonca@sakarya.edu.tr (Şükran Konca), mochidris@students.itb.ac.id (Mochammad Idris), hgunawan@math.itb.ac.id (Hendra Gunawan).
Peer review under responsibility of Egyptian Mathematical Society.

ELSEVIER Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.joems.2015.07.001

the usual 2-norm on ℓ^p. We also find that the 2-inner product $\langle ., . | . \rangle_{v,w}$ is actually defined on a larger space.

2. Definitions and preliminaries

Let X be a real vector space of dimension $d \geq 2$. The real-valued function $\langle \cdot, \cdot | \cdot \rangle$ which satisfies the following properties on X^3 is called 2-inner product on X, and the pair $(X, \langle \cdot, \cdot | \cdot \rangle)$ is called a 2-inner product space:

1. $\langle x, x | z \rangle \geq 0$; $\langle x, x|z \rangle = 0$ if and only if x and z are linearly dependent,
2. $\langle x, y|z \rangle = \langle y, x|z \rangle$,
3. $\langle x, x|z \rangle = \langle z, z|x \rangle$,
4. $\langle \alpha x, y|z \rangle = \alpha \langle x, y|z \rangle$, for $\alpha \in R$,
5. $\langle x_1 + x_2, y|z \rangle = \langle x_1, y|z \rangle + \langle x_2, y|z \rangle$.

The function $\|., .\|$: $X \times X \to [0, \infty)$, which follows four properties, is called a 2-norm and the pair $(X, \| \cdot, \cdot \|)$ is called a 2-normed space:

1. $\|x, z\| = 0$ if and only if x and z are linearly dependent,
2. $\|x, z\| = \|z, x\|$, for $x, z \in X$,
3. $\|\alpha x, z\| = |\alpha| \|x, z\|$, for $x, z \in X$ and $\alpha \in \mathbb{R}$,
4. $\|x + y, z\| \leq \|x, z\| + \|y, z\|$, for $x, y, z \in X$.

Definition 2.1. A sequence $(x^{(n)})$ in a linear 2-normed space $(X, \|., .\|)$ is called a Cauchy sequence, if for every y in X, $\lim_{n,m\to\infty} \left\| x^{(n)} - x^{(m)}, y \right\| = 0$, [3].

Definition 2.2. Let $\{a_1, a_2\}$ be a linearly independent set on a 2-normed space $(X, \|., .\|)$. A sequence $(x^{(n)})$ in X is called a Cauchy sequence with respect to the set $\{a_1, a_2\}$ if $\|x^{(n)} - x^{(m)}, a_1\| \to 0$ and $\|x^{(n)} - x^{(m)}, a_2\| \to 0$ as $m, n \to \infty$, [4].

Definition 2.3. A sequence $(x^{(n)})$ in a linear 2-normed space $(X, \|., .\|)$ is called a convergent sequence, if there is an x in X such that for every y in X, $\lim_{n\to\infty} \left\| x^{(n)} - x, y \right\| = 0$, [4].

Definition 2.4. Let $\{a_1, a_2\}$ be a linearly independent set on a 2-normed space $(X, \|., .\|)$. A sequence $(x^{(n)})$ in X is called a convergent sequence with respect to the set $\{a_1, a_2\}$ if there exists an $x \in X$ such that $\left\| x^{(n)} - x, a_1 \right\| \to 0$ and $\left\| x^{(n)} - x, a_2 \right\| \to 0$ as $n \to \infty$.

Definitions 2.1 and 2.3 are clearly stronger than Definitions 2.2 and 2.4. But in some cases, like in finite dimensional case and the standard case the respective two definitions are equivalent, which is not clear in the infinite dimensional case. But from the results in [5], we understand that the respective two definitions are still equivalent in the spaces ℓ^p and L^p (the space of p-integrable functions).

As we work with sequence spaces of real numbers, we will use the sum notation $\sum_k$ instead of $\sum_{k=1}^{\infty}$, for brevity. Throughout the paper we will use the following inequalities, (see, [6]):

$$\sum_k |a_k + b_k|^p \leq \sum_k |a_k|^p + \sum_k |b_k|^p \quad (0 < p \leq 1). \tag{2.1}$$

$$\left(\sum_{k=1}^{n} |a_k| \right)^p \leq n^{p-1} \sum_{k=1}^{n} |a_k|^p \quad (p \geq 1). \tag{2.2}$$

Note: The inequality (2.2) is a case of Hölder's inequality in the finite dimensional.

3. Main results

In this section, we begin with observation on ℓ^p, $1 \leq p < \infty$. It is well known that there exists $x \in \ell^q$ but $x \in \ell^p$ while $1 \leq p < q \leq \infty$. As sets, we have $\ell^p \subseteq \ell^q$ and the inclusion is strict. So, for $1 \leq p < q$, ℓ^p can actually be considered as a subspace of ℓ^q. For $q = 2$, we know that the norm $\| \cdot \|_2$ satisfies the parallelogram law. Then we can equip ℓ^p, $1 \leq p < 2$, with $\| \cdot \|_2$, so that ℓ^p became an inner product space with the inner product

$$\langle x, y \rangle := \sum_k x_k y_k.$$

Similarly, we realize that ℓ^p is a subspace of $(\ell^2, \|., .\|_2)$. Here, ℓ^p can be equipped with the 2-inner product

$$\langle x, y|z \rangle := \frac{1}{2} \sum_{k_1} \sum_{k_2} \begin{vmatrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2} \end{vmatrix} \begin{vmatrix} y_{k_1} & y_{k_2} \\ z_{k_1} & z_{k_2} \end{vmatrix}$$

and the 2-norm

$$\|x, z\|_2 : = \left(\frac{1}{2} \sum_{k_1} \sum_{k_2} \begin{vmatrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2} \end{vmatrix}^2 \right)^{\frac{1}{2}}.$$

We can check that the 2-norm $\|., .\|_2$ satisfies the parallelogram law:

$$\|x + y, z\|_2^2 + \|x - y, z\|_2^2 = 2\|x, z\|_2^2 + 2\|y, z\|_2^2$$

for every $x, y, z \in \ell^p$.

Next, we work on ℓ^p, $2 < p < \infty$. We note that the space ℓ^p is now larger than ℓ^2. Consequently, the usual 2-inner product and 2-norm on ℓ^2 are not used for all sequences in ℓ^p. Here, we present a new definition of 2-inner product and 2-norm on ℓ^p, which satisfies the parallelogram law, using weights.

For arbitrary $v = (v_k) \in \ell^p$, $v_k > 0$ $(\forall k \in \mathbb{N})$, set of ℓ_v^2 is defined by

$$\ell_v^2 := \left\{ x = (x_k) : \sum_k v_k^{p-2} x_k^2 < \infty, \ v = (v_k) \in \ell^p, \ v_k > 0 \ (\forall k \in \mathbb{N}) \right\}.$$

As set, we observe $\ell^p \subset \ell_v^2$ and the inclusion is strict. It is also known that v is not unique. Thus we have V_p, the collection of all sequences $v = (v_k) \in \ell^p$ where $v_k > 0$ for every $k \in \mathbb{N}$. Let v, w be in V_p, then v and w are *equivalent*, write $v \sim w$, if and only if there exists a constant $C > 0$ such that

$$\frac{1}{C} v_k \leq w_k \leq C v_k$$

for every $k \in \mathbb{N}$.

Let $v \sim w$ and $\begin{vmatrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2} \end{vmatrix} \neq 0$ if $k_1 \neq k_2$. Next we define the mapping which maps every triple of sequences $x = (x_k)$, $y = (y_k)$ and $z = (z_k)$ from ℓ^p to

$$\langle x, y|z \rangle_{v,w} := \frac{1}{2} \sum_{k_1} \sum_{k_2} \left\| \begin{matrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2} \end{matrix} \right\|^{p-2} \begin{vmatrix} x_{k_1} & z_{k_1} \\ x_{k_2} & z_{k_2} \end{vmatrix} \begin{vmatrix} y_{k_1} & z_{k_1} \\ y_{k_2} & z_{k_2} \end{vmatrix} \tag{3.1}$$