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Original Article

Geometric visualization of parallel bivariate Pareto distribution surfaces

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Abstract In the present paper, the differential-geometrical framework for parallel bivariate Pareto distribution surfaces $(P,\overline{P})$ is given. Curvatures of a curve lying on $(P,\overline{P})$, are interpreted in terms of the parameters of P. Geometrical and statistical interpretations of some results are introduced and plotted.

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1. Introduction

Information geometry (Geometry and Nature) has emerged from the study of invariant properties of the manifold of probability distributions. It is regarded as mathematical sciences having vast developing areas of applications as well as giving new trends in geometrical and topological methods. Information geometry has many applications which are treated in many different branches, for instance, statistical inference, linear and nonlinear systems, time series, neural networks, linear programing, convex analysis, completely integrable dynamical systems, quantum information geometry and geometric modeling [1]. A classical and intuitive way of describing the relationship between the differential geometry and the statistics is introduced, see, for instance [2–7], but in a slightly modified manner.

Pareto distribution is named after an Italian-born Swiss professor of economics, Vilfredo Pareto (1848–1923). Pareto [8] originally used this distribution to describe the allocation of wealth among individuals since it seemed to show rather well the way that a large portion of wealth of any society is owned by a smaller percentage of the people in that society [8,9]. Pareto distribution plays an important role in socio-economic studies. It is often used as a model for analyzing areas including city population distribution, stock price fluctuations and oil field location. In addition, it has found applications in the military area. It has been found to be suitable for approximating the right tail of distribution with positive skewness [10].

Bivariate Pareto distributions are popular models in many applied areas. They are very versatile and a variety of uncertainties can be usefully modeled by them. We mention: modeling of radiation carcinogenesis, performance measures for general sys-

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tems, reliability, modeling of drought, modeling of dependent heavy tailed risks with a non-zero probability of simultaneous loss and modeling of daily exchange rate data [11].

Creation of parallel surfaces is useful in design and manufacture. Enhancing or reducing the size of free-form surfaces requires calculation of curvature and other properties of a new surface, which is parallel to the original surface. In the Riemannian framework, several authors studied parallel and semi-parallel submanifolds, and a good survey can be found in [12].

In the differential geometry of surfaces, a Darboux frame is a natural moving frame constructed on a surface. It is the analog of the Frenet–Serret frame as applied to surface geometry. A geodesic curve is intrinsic to the geometric characterization of surfaces. Geodesics are used in many fields, for example, they are used in object segmentation, multi-scale image analysis, computer vision and image processing [13].

Abdel-All et al. [14] defined the parameter space of one-dimensional Pareto distribution of the first kind using its Fisher's matrix. They calculated the Riemannian and scalar curvatures to the parameter space. The differential equations of the geodesics are obtained and solved. The J-divergence, the geodesic distance and the relations between them are found. A development of the relation between the J-divergence and the geodesic distance is illustrated. The scalar curvature of the J-space is represented.

Many different forms of bivariate Pareto distributions have been constructed in the literature [15]. The main objective of this paper is to study a bivariate Pareto distribution (two-dimensional Pareto distribution) of the first kind that was given by Mardia, cited in [15], corresponding to the one-dimensional Pareto distribution of the first kind [14], without using its Fisher's matrix.

2. Geometrical and statistical preliminaries

Let $P : \mathbf{M} = \mathbf{M}(u,v)$ be an orientable surface and let $\mathbf{N}$ be a unit normal vector field of P. We consider a surface $\overline{P}$ to be parallel to P if there is a normal geodesic congruence between P and $\overline{P}$ such that the distance between corresponding points is constant, i.e. for each $\mathbf{M} \in P$ we have

$$\overline{P} : \overline{\mathbf{M}}(u,v) = \mathbf{M}(u,v) + r\mathbf{N}(u,v), \tag{1}$$

where, $r \neq 0$ is a real constant. We can say that P and $\overline{P}$ are parallel surfaces at distance r. If K,H and $\overline{K},\overline{H}$ denote the Gaussian and mean curvatures of P and $\overline{P}$, respectively, then we have [16]:

$$\overline{K} = \frac{K}{\Omega}, \quad \overline{H} = \frac{H + rK}{\Omega}, \quad \Omega = 1 + 2rH + r^2K \neq 0, \tag{2}$$

where, the relation between the principal curvatures (κ_1,κ_2) and $(\overline{\kappa}_1,\overline{\kappa}_2)$ of $(P,\overline{P})$ is given by

$$\overline{\kappa}_1 = \frac{\kappa_1}{1 + r\kappa_1}, \quad \overline{\kappa}_2 = \frac{\kappa_2}{1 + r\kappa_2}.$$

Let P be a surface, and let $\boldsymbol{\beta}$ be a unit speed curve on P. At each point on $\boldsymbol{\beta}$, consider the following three vectors: the unit normal vector $\mathbf{N}$ to the surface, the unit tangent vector $\mathbf{t}$ to the curve and the tangent normal vector $\mathbf{E} = \mathbf{N} \wedge \mathbf{t}$. This vector is tangent to the surface P, but normal to the curve $\boldsymbol{\beta}$. These vectors $\{\mathbf{t};\mathbf{E};\mathbf{N}\}$ form a right-handed frame, known as the Darboux frame for $\boldsymbol{\beta}$ on P. Darboux equations for this frame are given by [16,17]

$$\frac{d}{ds}\begin{pmatrix}\mathbf{t}\\ \mathbf{E}\\ \mathbf{N}\end{pmatrix} = \begin{pmatrix}0 & \kappa_g & \kappa_n\\ -\kappa_g & 0 & \tau_g\\ -\kappa_n & -\tau_g & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{E}\\ \mathbf{N}\end{pmatrix}, \tag{3}$$

where κ_g is the geodesic curvature, κ_n is the normal curvature and τ_g is the geodesic torsion of $\boldsymbol{\beta}$. Thus, we can write κ_g, κ_n and τ_g in the form

$$\kappa_g = (\boldsymbol{\beta}',\mathbf{N},\boldsymbol{\beta}''), \quad \kappa_n = (\boldsymbol{\beta}'',\mathbf{N}), \quad \tau_g = (\boldsymbol{\beta}',\mathbf{N},\mathbf{N}'), \tag{4}$$

and if $\boldsymbol{\beta}$ is not parameterized by arc length, the above relations take the forms

$$\kappa_g = \frac{1}{|\boldsymbol{\beta}'|^3}(\boldsymbol{\beta}',\mathbf{N},\boldsymbol{\beta}''), \quad \kappa_n = \frac{1}{|\boldsymbol{\beta}'|^2}(\boldsymbol{\beta}'',\mathbf{N}), \quad \tau_g = \frac{1}{|\boldsymbol{\beta}'|}(\boldsymbol{\beta}',\mathbf{N},\mathbf{N}'). \tag{5}$$

The bivariate distribution with joint density function for $\alpha > 0$

$$f_{X,Y}\ (x,y;\gamma,\sigma,\alpha) = \alpha(\alpha+1)(\gamma\sigma)^{\alpha+1}\lambda^{-(\alpha+2)}, \quad x \geq \gamma > 0,\ y \geq \sigma > 0, \tag{6}$$

where, $\lambda = \sigma x + \gamma y - \gamma\sigma$ may be called a bivariate Pareto distribution of the first kind [15], since the marginal distributions have density functions

$$f_{X_i}(x_i;\theta_i,\alpha) = \alpha\,\theta_i^{\alpha}\,x_i^{-(\alpha+1)}, \quad x_i \geq \theta_i > 0,\ i = 1,2, \tag{7}$$

where, $X_1 = X, X_2 = Y, x_1 = x, x_2 = y, \theta_1 = \gamma, \theta_2 = \sigma$.

It can be seen that, for $\alpha > 1, \alpha > 2$,

$$E(X_i) = \frac{\alpha}{\alpha-1}\theta_i, \quad E(X_1X_2) = \frac{(\alpha^2-\alpha-1)}{(\alpha-1)(\alpha-2)}\theta_1\theta_2,$$
$$Var(X_i) = \frac{\alpha}{(\alpha-1)^2(\alpha-2)}\theta_i^2. \tag{8}$$

The conditional density function of Y, given $X = x$, is

$$f_{Y|X}\ (y|x) = (\alpha+1)\gamma(\sigma x)^{\alpha+1}\lambda^{-(\alpha+2)}, y \geq \sigma > 0, \gamma > 0, \alpha > 0. \tag{9}$$

The conditional density function of X, given $Y = y$, is

$$f_{X|Y}\ (x|y) = (\alpha+1)\sigma(\gamma y)^{\alpha+1}\lambda^{-(\alpha+2)}, \quad x \geq \gamma > 0, \sigma > 0, \alpha > 0. \tag{10}$$

Therefore, for $\alpha > 1$, we also find

$$E(Y|X = x) = \sigma\left(1 + \frac{x}{\gamma\alpha}\right),$$
$$Var(Y|X = x) = \left(\frac{\sigma}{\gamma}\right)^2\frac{(\alpha+1)x^2}{\alpha^2(\alpha-1)}, \tag{11}$$

$$E(X|Y = y) = \gamma\left(1 + \frac{y}{\sigma\alpha}\right),$$
$$Var(X|Y = y) = \left(\frac{\gamma}{\sigma}\right)^2\frac{(\alpha+1)y^2}{\alpha^2(\alpha-1)}. \tag{12}$$

Using (8), we find

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{\gamma\sigma}{(\alpha-1)^2(\alpha-2)}, \quad \alpha \neq 1, \alpha \neq 2, \tag{13}$$

and consequently, the correlation between X and Y, denoted by $R \equiv Cor(X,Y)$, is given from