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Original Article

Totally real submanifolds of Kaehler product manifolds

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Abstract Totally real submanifolds have been studied by many geometers in different ambient manifolds. The purpose of this note is to study totally real submanifolds in Kaehlerian product manifolds. We derive some integral formulas computing the Laplacian of the square of the second fundamental form and using these formulas we prove pinching theorems. In fact, we have generalized some results due to Yano and Kon [1,2] to the case when the ambient manifold is Kaehlerian product manifold.

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1. Introduction

The geometry of totally real submanifolds is an interesting field which is studied by many geometers. For example, Houh [3], Yau [4], Chen and Ogiue [5] have studied totally real submanifolds in an almost Hermitian manifold or a Kaehlerian manifold of constant holomorphic sectional curvature and obtained many interesting results. Moreover, Yano and Kon [1,2] have generalized some of the results proved in [6–9]. On the other hand, Kaehlerian product manifold has also been paid attention by geometers [10]. The object of this note is to study the geometry of totally real submanifolds when the ambient manifold is a Kaehlerian product manifold.

2. Preliminaries

Let $\overline{M}^n$ be a Kaehlerian manifold of complex dimension n (of real dimension $2n$) and $\overline{M}^p$ be a Kaehlerian manifold of complex dimension p (of real dimension $2p$). Let us denote by J_n and J_p almost complex structures of $\overline{M}^n$ and $\overline{M}^p$ respectively. Now, we suppose that $\overline{M}^n$ and $\overline{M}^p$ are complex space forms with constant holomorphic sectional curvatures c_1 and c_2 and denote them by $\overline{M}^n(c_1)$ and $\overline{M}^p(c_2)$ respectively. The Riemannian curvature tensor $\overline{R}_n$ of $\overline{M}^n(c_1)$ is given by

$$\overline{R}_n(X,Y)Z = \frac{1}{4}c_1[g_n(Y,Z)X - g_n(X,Z)Y] + \frac{1}{4}c_1[g_n(J_nY,Z)J_nX - g_n(J_nX,Z)J_nY + 2g_n(X,J_nY)J_nZ]$$

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and the Riemannian curvature tensor $\overline{R}_p$ of $\overline{M}^p(c_2)$ is given by

$$\overline{R}_p(X,Y)Z = \frac{1}{4}c_2[g_p(Y,Z)X - g_p(X,Z)Y] + \frac{1}{4}c_2[g_p(J_pY,Z)J_pX - g_p(J_pX,Z)J_pY + 2g_p(X,J_pY)J_pZ].$$

We consider the Kaehlerian product manifold $\overline{M} = \overline{M}^n(c_1) \times \overline{M}^p(c_2)$. Let us denote by P and Q the projection operators of the tangent space of $\overline{M}^n(c_1)$ and $\overline{M}^p(c_2)$ respectively. Then, we have $P^2 = P, Q^2 = Q, PQ = QP = 0$. We put $F = P - Q$ and it can be verified that $F^2 = I$. Thus, F is almost product structure on $\overline{M}$. Moreover, we define a Riemannian metric g on $\overline{M}$ by

$$g(X,Y) = g_n(PX,PY) + g_p(QX,QY)$$

for any vector field X and Y of $\overline{M}$. It also follows that $g(FX,Y) = g(FY,X)$. Let us put $JX = J_nPX + J_pQX$ for any vector field X of $\overline{M}$. Then we see that $J_nP = PJ, J_pQ = QJ, FJ = JF, J^2 = -I, g(JX,JY) = g(X,Y), \overline{\nabla}_X J = 0$. Thus, J is Kaehlerian structure on $\overline{M}$. The Riemannian curvature tensor $\overline{R}$ of a Kaehlerian product manifold $\overline{M}$ is given by [10]

$$\begin{aligned}\overline{R}(X,Y,Z,W) = {} & \frac{1}{16}(c_1 + c_2)[g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ & + g(JY,Z)g(JX,W) - g(JX,Z)g(JY,W) \\ & + 2g(X,JY)g(JZ,W) + 2g(FY,Z)g(FX,W) \\ & - g(FX,Z)g(FY,W) + g(FJY,Z)g(FJX,W) \\ & - g(FJX,Z)g(FJY,W) + 2g(FX,JY)g(FJZ,W)] \\ & + \frac{1}{16}(c_1 - c_2)[g(FY,Z)g(X,W) - g(FX,Z)g(Y,W) \\ & + g(Y,Z)g(FX,W) - g(X,Z)g(FY,W) \\ & + g(FJY,Z)g(JX,W) - g(FJX,Z)g(JY,W) \\ & + g(JY,Z)g(FJX,W) - g(JX,Z)g(FJY,W) \\ & + 2g(FX,JY)g(JZ,W) \\ & + 2g(X,JY)g(JFZ,W)]\end{aligned} \tag{2.1}$$

for any vector fields X, Y and Z on $\overline{M}$. An n-dimensional Riemannian manifold M isometrically immersed in a Kaehlerian product manifold $\overline{M}$ is called totally real submanifold of $\overline{M}$ if $JT_x(M) \perp T_x(M)$ for each $x \in M$ where $T_x(M)$ denotes the tangent space to M at $x \in M$. Here we have identified $T_x(M)$ with its image under the differential of the immersion because our computation is local. If $X \in T_x(M)$, then JX is a normal vector to M. Let $\overline{g}$ be the metric tensor field of $\overline{M}$ and g be the induced metric tensor field on M. We denote by $\overline{\nabla}$ (resp. ∇) the operator of covariant differentiation with respect to $\overline{g}$ (resp. g). Then the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y) \tag{2.2}$$

$$\overline{\nabla}_X N = -A_N X + D_X N \tag{2.3}$$

for any tangent vector fields X, Y and normal vector field N on M, where D is the operator of covariant differentiation with respect to the linear connection induced in the normal bundle. Both A and B are called the second fundamental form of M and satisfy

$$\overline{g}(B(X, Y), N) = g(A_N X, Y). \tag{2.4}$$

A normal vector field N in the normal bundle is said to be parallel if $D_X N = 0$ for any tangent vector field X on M. The mean curvature vector H is defined as $H = (1/n)TrB$, where $TrB = \sum_i B(e_i, e_i)$ for an orthonormal frame $\{e_i\}$. We say that

- M is minimal if $H = 0$.
- M is totally umbilical if the second fundamental form of M satisfies $B(X, Y) = g(X, Y)H$.
- M is totally geodesic if the second fundamental form of M vanishes identically, that is, $B = 0$.

We choose a local field of orthonormal frames $e_1, \ldots, e_n;\ e_{n+1}, \ldots, e_{n+p};\ e_{1^*} = Je_1, \ldots, e_{n^*} = Je_n;\ e_{(n+1)^*} = Je_{n+1}, \ldots, e_{(n+p)^*} = Je_{n+p}$ in $\overline{M}$ in such a way that restricted to $M, e_1, \ldots, e_n$ are tangent to M. With respect to this frame field of $\overline{M}$, let $\omega^1, \ldots, \omega^n;\ \omega^{n+1}, \ldots, \omega^{n+p};\ \omega^{1^*}, \ldots, \omega^{n^*};\ \omega^{(n+1)^*}, \ldots, \omega^{(n+p)^*}$ be the field of dual frames. Unless otherwise stated, we use the conventions that the ranges of indices are respectively $A, B, C, D = 1, \ldots, n + p, 1^*, \ldots, (n + p)^*$; $i, j, k, l, t, s = 1, \ldots, n$; $a, b, c, d = n + 1, \ldots, n + p, 1^*, \ldots, (n + p)^*$; $\alpha, \beta, \gamma = n + 1, \ldots, n + p$; $\lambda, \mu, \nu, = n + 1, \ldots, n + p, (n + 1)^*, \ldots, (n + p)^*$ and that when an index appears twice in any term as a subscript and a superscript, it is understood that this index is summed over its range. Then the structure equations of $\overline{M}$ are given by

$$\begin{aligned}& d\omega^A = -\omega^A_B\omega^B, \quad \omega^A_B + \omega^B_A = 0, \\ & \omega^i_j + \omega^j_i = 0, \quad \omega^i_j = \omega^{i^*}_{j^*}, \quad \omega^{i^*}_j = \omega^{j^*}_i, \\ & \omega^\alpha_\beta + \omega^\beta_\alpha = 0, \quad \omega^\alpha_\beta = \omega^{\alpha^*}_{\beta^*}, \quad \omega^{\alpha^*}_\beta = \omega^{\beta^*}_\alpha,\end{aligned} \tag{2.5}$$

$$\begin{aligned}& \omega^i_\alpha + \omega^\alpha_i = 0, \quad \omega^i_\alpha = \omega^{i^*}_{\alpha^*}, \quad \omega^{i^*}_\alpha = \omega^{\alpha^*}_i, \\ & d\omega^A_B = -\omega^A_C\omega^C_B + \phi^A_B, \quad \phi^A_B = \frac{1}{2}K^A_{BCD}\omega^C \wedge \omega^D\end{aligned} \tag{2.6}$$

Restricting these forms to M, we have

$$\omega^a = 0, \tag{2.7}$$

$$d\omega^i = -\omega^i_k \wedge \omega^k, \tag{2.8}$$

$$d\omega^i_j = -\omega^i_k \wedge \omega^k_j + \Omega^i_j, \quad \Omega^i_j = \frac{1}{2}R^i_{jkl}\omega^k \wedge \omega^l \tag{2.9}$$

Since $0 = d\omega^a = -\omega^a_i \wedge \omega^i$, by Cartan's lemma we have

$$\omega^a_i = h^a_{ij}\omega^j, \quad h^a_{ij} = h^a_{ji} \tag{2.10}$$

We see that $g(A_a e_i, e_j) = h^a_{ij}$. The Gauss-equation is given by

$$R^i_{jkl} = K^i_{jkl} + \sum_a (h^a_{ik}h^a_{jl} - h^a_{il}h^a_{jk}) \tag{2.11}$$

Moreover we have

$$d\omega^a_b = -\omega^a_c \wedge \omega^c_b + \Omega^a_b, \quad \Omega^a_b = \frac{1}{2}R^a_{bkl}\omega^k \wedge \omega^l \tag{2.12}$$

and the Ricci-equation is given by

$$R^a_{bkl} = K^a_{bkl} + \sum_i (h^a_{ik}h^b_{il} - h^a_{il}h^b_{ik}) \tag{2.13}$$

From (2.5) and (2.10) we have

$$h^{i^*}_{jk} = h^{j^*}_{ik} = h^{k^*}_{ij} \tag{2.14}$$

We define the covariant derivative h^a_{ijk} of h^a_{ij} by setting