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Original Article

Classification of conics and Cassini curves in Minkowski space-time plane

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Abstract In this paper we use the Apollonius definition of conics to generate algebraic curves in the Minkowski space-time plane M^2, which turn out to be different from classical conic sections. We extend and classify this sort of "M-conics". We discuss the cases of the singularity points of these M-conics, coming from the transition from timelike world to spacelike world through the lightlike one. Finally, we translate the classical concept of Cassini curves with two foci and that of (multifocal) Cassini curves to Minkowski planes M^2.

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1. Introduction

A Minkowski space-time plane M^2 is pseudo-Euclidean plane, i.e., there are three types of directions, the spacelike, timelike and lightlike directions, and the unit ball in such a plane consists of two conjugate hyperbolas with lightlike asymptotes [1,2], see Fig. 1. Many authors discuss this space from the relativity point of view with some mathematical concepts, e.g., Naber [3–5].

In the following we use the fundamental Apollonian definitions of quadratic conics in Euclidean plane to define "M-conics" in the Minkowski space-time plane M^2.

The elementary geometric Apollonius definition of an ellipse in the Euclidean plane reads as follows:

> *An ellipse is the set of points P having constant distance sum from two fixed points F_1, F_2, the so-called focal points of the ellipse.*

Similar and well-known definitions exist for hyperbolas and parabolas. While the projective geometric point of view distinguishes these "conic sections" (or shortly "conics") by their ideal points, a proper elementary geometric approach has to omit this way to classify conics.

Cassini's modification [6–8], in 1680, of the Apollonius definition replaces the constant sum of distances to two fixed focal points by the constant product of these distances. Cassini believed that the motion of the planets of the solar system revolves in one of these curves. There are many applications of Cassini

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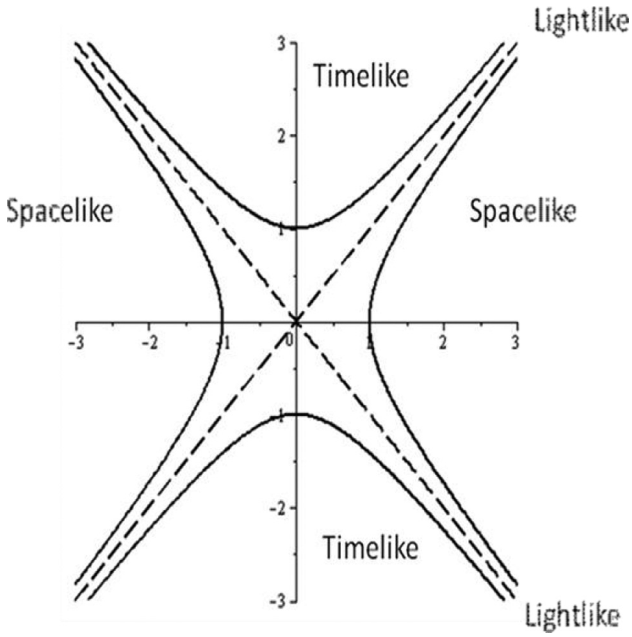

Fig. 1 Unit circle S_M of the Minkowski plane M^2.

curves in BioGeometry, e.g. sectors of onions layers, bacterial colonies and cell shapes. Furthermore, simulation of light scattering [9], by small concave particles is necessary to find an appropriate mathematical description of the particle shape. This can be done easily by the usage of Cassini curves. For example, this approach is used to fit the shape of the human red blood cell [10,11]. For more applications of Cassini curves, see [12–14].

In this paper we aim at changing the place of action from the classical Euclidean plane to the so-called Minkowski space-time plane or, physically spoken, to the two-dimensional space-time world. As expected the topology of the "M-conics" and "M-Cassini curves" depends on the position and distance of the focal points and of the chosen constant distance sum resp. product. Special cases occur, if these foci lie on a lightlike line.

2. Minkowski norm

The Lorentz transformations are designed to preserve the lightlike lines, which is "M-circle true", i.e. they are translations together with pseudo-reflections and pseudo-rotations, which is the set $x^2 - y^2 = 0$. In fact, it preserves each of the hyperbolas $x^2 - y^2 = k$, for all k, i.e. the common asymptotes of these hyperbolas are the lightlike directions.

Definition 1. The Minkowski space-time plane M^2 is a real vector space with usual Minkowski inner product $\langle\ ,\ \rangle_M$ given by

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle_M := x_1y_1 - x_2y_2, \tag{1}$$

where $\boldsymbol{x}, \boldsymbol{y} \in M^2$, $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$.

The norm $\|\cdot\|$ is defined by the previous inner product as

$$\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x}\rangle_M|}. \tag{2}$$

Any arbitrary vector $\boldsymbol{x} \in M^2$ is classified according to the sign of $\langle \boldsymbol{x}, \boldsymbol{x}\rangle_M$ as follows:

i $\boldsymbol{x}$ is timelike if $\langle \boldsymbol{x}, \boldsymbol{x}\rangle_M < 0$,
ii $\boldsymbol{x}$ is spacelike if $\langle \boldsymbol{x}, \boldsymbol{x}\rangle_M > 0$,
iii $\boldsymbol{x}$ is lightlike if $\langle \boldsymbol{x}, \boldsymbol{x}\rangle_M = 0$.

It is clear to define the "unit circle" as a pair of Euclidean hyperbolas $x^2 - y^2 = \pm 1$ as follows:

$$S_M := \{\boldsymbol{x} \in M^2 : \|\boldsymbol{x}\| = 1\}, \tag{3}$$

and the unit ball is

$$B_M := \{\boldsymbol{x} \in M^2 : \|\boldsymbol{x}\| \le 1\}. \tag{4}$$

The unit circle S_M has four sheets. Two of them come from the equation $\langle \boldsymbol{x}, \boldsymbol{x}\rangle_M = 1$ which refers to the spacelike directions. The others from $\langle \boldsymbol{x}, \boldsymbol{x}\rangle_M = -1$, which refers to the timelike directions. Any lightlike vector is parallel to the asymptotes $\boldsymbol{y} = \boldsymbol{x}$ and $\boldsymbol{y} = -\boldsymbol{x}$ of the unit circle S_M, see Fig. 1. The pair of asymptotes of S_M forms the so-called light cone of M^2.

3. Conics in Minkowski space-time planes M^2

3.1. M-ellipse in M^2

We discuss the conics in Minkowski plane M^2 by using the usual definition like in the Euclidean plane. Using the metric definition of M^2, an M-ellipse obtains by distance sum from its foci $\boldsymbol{z}$ and $\boldsymbol{w}$ to the locus point $\boldsymbol{x}$ on it:

$$\|\boldsymbol{x} - \boldsymbol{z}\| + \|\boldsymbol{x} - \boldsymbol{w}\| = 2a,\ a > 0 \tag{5}$$

where $\boldsymbol{x} = (x, y)$ is a position point with given two foci points $\boldsymbol{z} = (z_1, z_2)$ and $\boldsymbol{w} = (w_1, w_2)$.

3.2. Classification of M-ellipse in M^2

Now, we classify all cases of the M-ellipse (5). First of all, we can rewrite (5) as follows:

$$\sqrt{\left|(x - z_1)^2 - (y - z_2)^2\right|} + \sqrt{\left|(x - w_1)^2 - (y - w_2)^2\right|} = 2a. \tag{6}$$

Generally, M-ellipses (6) have at most eight singular points. Furthermore, they are symmetric with respect to the center of their focal segments. Denote the distance between the two foci $\boldsymbol{z} = (z_1, z_2)$ and $\boldsymbol{w} = (w_1, w_2)$ by $2d$. Then we have

$$d = \frac{1}{2}\sqrt{\left|(z_1 - w_1)^2 - (z_2 - w_2)^2\right|}. \tag{7}$$

To analyze the singularity of (6), we need to find the limit points of (6) with the four lines $y = x + z_2 - z_1$, $y = -x + z_2 + z_1$, $y = x + w_2 - w_1$ and $y = -x + w_2 + w_1$.

Case I:

In the region of $|x - z_1| \ge |y - z_2|$ and $|x - w_1| \ge |y - w_2|$ we have two symmetric parts with $\sqrt{(x - z_1)^2 - (y - z_2)^2} + \sqrt{(x - w_1)^2 - (y - w_2)^2} = 2a$. We have the following singular points on the four lines $y = x + z_2 - z_1$, $y = -x + z_2 + z_1$, $y = x + w_2 - w_1$ and $y = -x + w_2 + w_1$.

1- The limit point $P_1 = (x_1, y_1)$ at the line $y = x + z_2 - z_1$; therefore we have

$$x_1 = \frac{4a^2 + w_2^2 - (z_1 - z_2 - w_1)^2}{2(z_1 - z_2 - w_1 + w_2)} + z_1 - z_2, \tag{8}$$