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Generalization of Herstein theorem and its applications to range inclusion problems ☆

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Abstract Let R be an associative ring. An additive mapping $d : R \to R$ is called a Jordan derivation if $d(x^2) = d(x)x + xd(x)$ holds for all $x \in R$. The objective of the present paper is to characterize a prime ring R which admits Jordan derivations d and g such that $[d(x^m), g(y^n)] = 0$ for all $x, y \in R$ or $d(x^m) \circ g(y^n) = 0$ for all $x, y \in R$, where $m \geqslant 1$ and $n \geqslant 1$ are some fixed integers. This partially extended Herstein's result in [6, Theorem 2], to the case of (semi)prime ring involving pair of Jordan derivations. Finally, we apply these purely algebraic results to obtain a range inclusion result of continuous linear Jordan derivations on Banach algebras.

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1. Introduction

Throughout this paper R will denote an associative ring with center $Z(R)$. Recall that a ring R is said to be prime if for any $a, b \in R, aRb = \{0\}$ implies $a = 0$ or $b = 0$, and R is semiprime if for any $a \in R, aRa = \{0\}$ implies $a = 0$. A ring R is said to be n-torsion free, where $n > 1$ is an integer, in case $nx = 0$ implies $x = 0$ for all $x \in R$. For any $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$ and the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. Following [1], an additive mapping $d : R \to R$ is said to be a derivation (resp. Jordan derivation) on R if $d(xy) = d(x)y + xd(y)$ (resp. $d(x^2) = d(x)x + xd(x)$) holds for all $x, y \in R$. Let S be a nonempty subset of R. A mapping $f : R \to R$ is called centralizing on S if $[f(x), x] \in Z(R)$ for all $x \in S$ and is called commuting on S if $[f(x), x] = 0$ for all $x \in S$. The study of such mappings were initiated by Posner. In [2, Lemma 3], Posner proved that if a prime ring R has a nonzero commuting derivation on R, then R is commutative. This result was subsequently refined and extended by a number of algebraists; we refer the reader to [3–5] for a state-of-art account and a comprehensive bibliography.

In [6], Herstein proved the following result: If R is a prime ring of characteristic not two admitting a nonzero derivation d such that $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative. Further, Daif [7] showed that a 2-torsion free semiprime

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ring R admits a derivation d such that $[d(x), d(y)] = 0$ for all $x, y \in I$, where I is a nonzero ideal of R and d is nonzero on I, then R contains a nonzero central ideal. Motivated by the above result, Ashraf and Rehman [8] proved that if R is a 2-torsion free prime ring admitting a nonzero derivation d such that $d(x) \circ d(y) = 0$ for all $x, y \in I$, where I is a nonzero ideal of R, then R is commutative. This result was further extended by first author together with Shuliang [9, Theorem 3.2] for semiprime rings. In Section 3, our aim is to generalize these results for pair of Jordan derivations d and g. More precisely, it was shown that if R is a $\max\{m, n, 2\}!$-torsion free prime ring, where $m \geqslant 1$ and $n \geqslant 1$ are some fixed integers, and d, g are nonzero Jordan derivations of R such that $[d(x^m), g(y^n)] = 0$ for all $x, y \in R$, then R is commutative. Further, some more related results have also been discussed. In Section 4, we apply purely algebraic results from Section 3 to discuss the range inclusion problems in the setting of continuous linear Jordan derivations on Banach algebras. Throughout this paper, we assume that $m \geqslant 1$ and $n \geqslant 1$ are some fixed integers.

2. Some preliminaries

We shall do a great deal of calculations with commutators and anti-commutators, routinely using the following basic identities: For all $x, y, z \in R$;

$$[xy, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = [x, y]z + y[x, z]$$
$$(x + y) \circ z = x \circ z + y \circ z \text{ and } x \circ (y + z) = x \circ y + x \circ z$$
$$x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$$
$$(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z].$$

We begin with the following lemmas which are essential for developing the proof of our results.

Lemma 2.1 [4, Theorem 4]. *Let R be a prime ring and I a nonzero left ideal of R. If R admits a nonzero derivation d which is centralizing on I, then R is commutative.*

Lemma 2.2 [10, Lemma 4]. *Let R be a 2-torsion free semiprime ring and $a, b \in R$. If for all $x \in R$ the relation $axb + bxa = 0$ holds, then $axb = bxa = 0$ is fulfilled for all $x \in R$.*

Lemma 2.3 [11, Lemma 1]. *Let R be an $m!$-torsion free ring. Suppose $y_1, y_2, \ldots, y_m \in R$ satisfying $\alpha y_1 + \alpha^2 y_2 + \ldots + \alpha^m y_m = 0$ for $\alpha = 1, 2, \ldots, m$. Then $y_i = 0$ for all i.*

Lemma 2.4 [12, Lemma 3.2]. *A continuous Jordan derivation on a Banach algebra leaves invariant the primitive ideals in the algebra.*

3. Generalizations of the condition $d(x)d(y) = d(y)d(x)$

To state our results precisely, we fix some notations. From now, Q always denotes the maximal right ring of quotients of R. If R is a (semi)prime ring, then Q is also a (semi)prime ring. The center of Q is called the extended centroid of R and is denoted by C. For the explanation of maximal right ring of quotients we refer the reader to [13]. We shall use the fact that any semiprime ring R and its maximal right ring of quotients Q satisfy the same differential identities which is very useful since Q contains the identity element (see Theorem 3 in [14]). For the explanation of differential identities we refer the reader to [15,16]. Throughout this section, we will use the fact that image of the identity of a ring R is zero under any derivation. We begin our investigations with the following theorem which generalizes Theorem 2 in [6].

Theorem 3.1. *Let R be a $\max\{m, n, 2\}!$-torsion free prime ring, and d, g be nonzero Jordan derivations of R. If $[d(x^m), g(y^n)] = 0$ holds for all $x, y \in R$, then R is commutative.*

Proof. Since d and g are Jordan derivations on R, d and g also are derivations on R by Herstein's theorem [1]. By the assumption, we have

$$[d(x^m), g(y^n)] = 0 \quad \text{for all } x, y \in R.$$

It is well known that R and Q satisfy the same differential identities [14, Theorem 3]. Therefore

$$[d(x^m), g(y^n)] = 0 \quad \text{for all } x, y \in Q. \tag{3.1}$$

Note that Q has the identity element. Replacing x by $1 + x$ in (3.1), we get

$$\binom{m}{1}[d(x), g(y^n)] + \binom{m}{2}[d(x^2), g(y^n)] + \cdots + \binom{m}{m}[d(x^m), g(y^n)] = 0. \tag{3.2}$$

Substituting px for x in (3.2), where $p = 1, 2, \ldots, m$, we get

$$p\binom{m}{1}[d(x), g(y^n)] + p^2\binom{m}{2}[d(x^2), g(y^n)] + \cdots + p^m\binom{m}{m}[d(x^m), g(y^n)] = 0.$$

Using Lemma 2.3, we obtain $\binom{m}{r}[d(x^r), g(y^n)] = 0$ for all $x, y \in Q$ and $r = 1, 2, \ldots, m$. In particular for $r = 1$, we have $m[d(x), g(y^n)] = 0$ for $x, y \in Q$. By applying torsion free fact of Q, we are forced to conclude that

$$[d(x), g(y^n)] = 0 \quad \text{for all } x, y \in Q.$$

Now, replacing y by $y + 1$ and using similar approach as above, we obtain

$$[d(x), g(y)] = 0 \quad \text{for all } x, y \in Q. \tag{3.3}$$

Again replace y by yz in (3.3) to get

$$[d(x), g(y)]z + g(y)[d(x), z] + y[d(x), g(z)] + [d(x), y]g(z) = 0 \quad \text{for all } x, y, z \in Q. \tag{3.4}$$

Application of (3.3) yields that

$$g(y)[d(x), z] + [d(x), y]g(z) = 0 \quad \text{for all } x, y, z \in Q. \tag{3.5}$$

Substituting rz for z in (3.5) and using it, we get

$$g(y)r[d(x), z] + [d(x), y]rg(z) = 0 \quad \text{for all } r, x, y, z \in Q.$$

In particular, for $y = z$, we have