Journal of the Egyptian Mathematical Society (2014) 22, 341–347

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Continuous and integrable solutions of a nonlinear Cauchy problem of fractional order with nonlocal conditions

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Received 11 August 2013; revised 25 November 2013; accepted 17 December 2013
Available online 6 February 2014

KEYWORDS

Nonlinear fractional problem;
Riemann–Liouville fractional derivative;
Weighted Cauchy problem;
Nonlocal condition;
Schauder fixed point theorem

Abstract In this article, we discuss the existence of at least one solution as well as uniqueness for a nonlinear fractional differential equation with weighted initial data and nonlocal conditions. The existence of at least one L_1 and continuous solution will be proved under the Caratheòdory conditions via a classical fixed point theorem of Schauder. An example is also given to illustrate the efficiency of the main theorems.

MATHEMATICS SUBJECT CLASSIFICATION: 26A33; 34K37; 34A08; 45E10; 47H10

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1. Introduction

Fractional differential equations have gained considerable importance due to their application in various sciences, such as physics, mechanics, chemistry, engineering, etc. In the recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives, see the monographs of Kilbas et al. [1], Miller and Ross [2], Podlubny [3], and the papers [4–16] and the references therein.

Let $I = (0, T], L_1 = L_1(0, T]$ be the space of Lebesgue integrable functions on I. and $C(0, T]$ be the space of continuous functions defined on I.

Consider the weighted nonlocal Cauchy type fractional problem

$$D^\alpha(p(t)u(t)) = f(t, u(t)) \quad a.e.\ t \in (0, T], T < \infty \tag{1}$$

$$\lim_{t\to 0^+} t^{1-\alpha}p(t)u(t) = \sum_{j=1}^{m} a_j u(\tau_j), \quad \tau_j \in (0, T). \tag{2}$$

where D^α denoted the Riemann–Liouville derivative of order $\alpha \in (0, 1]$.

Problems with non-local conditions have been extensively studied by several authors in the last two decades. The reader is referred to [7–9,17–19] and references therein.

Nonlinear fractional differential equation with weighted initial data has been carried out by various researchers. In present, there are some papers which deal with the existence and multiplicity of solutions for weighted nonlinear fractional differential equations.

In [12] Khaled et al. studied the weighted Cauchy-type problem

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Peer review under responsibility of Egyptian Mathematical Society.

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http://dx.doi.org/10.1016/j.joems.2013.12.008

$$(I)\begin{cases} D^{\alpha}u(t) = f(t,u), & t \in (0,T] \\ t^{1-\alpha}u(t)|_{t=0} = b, \end{cases}$$

where D^{α} is the fractional derivative (in the sense of Riemann–Liouville) of order $0 < \alpha < 1$, f is a continuous nonlinear function.

In [10] Furati et al. studied the weighted Cauchy-type problem (I) where $f(t,u)$ is assumed to be continuous on $R^+ \times R$ and $|f(t,u)| \leqslant t^{\mu}e^{-\sigma t}\psi(t)|u|^m$.

Also in [5] El-Sayed et al. studied the problem (I) where the function f satisfies Carathèodory conditions with growth condition. In [19] the existence and uniqueness of the solution of the problem (I) was discussed by using the method of upper and lower solutions and its associated monotone iterative.

In [16] Weia et al. studied the existence and uniqueness of the solution of the periodic boundary value problem for a fractional differential equation involving a Riemann–Liouville fractional derivative

$$\begin{cases} D^{\alpha}u(t) = f(t,u), & t \in (0,T] \\ t^{1-\alpha}u(t)|_{t=0} = t^{1-\alpha}u(t)|_{t=T}. \end{cases}$$

by using the monotone iterative method. In [11] Jankowski discussed the existence of solutions of fractional equations of Volterra type with the Riemann–Liouville derivative,

$$\begin{cases} D^{\alpha}x(t) = f(t, x(t), \int_0^t k(t,s)x(s)ds), & t \in (0,T] \\ t^{1-\alpha}u(t)|_{t=0} = r, \end{cases}$$

existence results are obtained by using a Banach fixed point theorem with weighted norms and by a monotone iterative method.

In [4] Belmekki et al. studied the existence and uniqueness of the solution for a class of fractional differential equations

$$\begin{cases} D^{\alpha}u(t) - \lambda u(t) = f(t, u(t)), & t \in (0,1] \\ \lim_{t\to 0^+} t^{1-\alpha}u(t) = u(1) \end{cases}$$

by using the fixed point theorem of Schaeffer and the Banach contraction principle.

In this paper we will study the existence of solutions for problem (1) and (2) with certain nonlinearities, using the equivalence of the fractional differ-integral problem with the corresponding Volterra integral equation. We prove the existence of at least L_1 and continuous solutions of the problem (1) and (2) such that the function f satisfies Carathèodory conditions and

$$|f(t,u)| \leqslant h(t), \quad \text{a.e } t \in (0,T]$$

where $h(t)$ is a Lebesgue function on $(0,T]$. Also the uniqueness of the solution will be studied.

Our problem (1) and (2) includes as a special case when $p(t) = 1$, the nonlocal fractional differential equation

$$D^{\alpha}u(t) = f(t,u(t)) \quad a.e.\ t \in (0,T], T < \infty$$

$$\lim_{t\to 0^+} t^{1-\alpha}\ u(t) = \sum_{j=1}^{m} a_j u(\tau_j), \quad \tau_j \in (0,T).$$

2. Preliminaries

In this section, we present some definitions, lemmas and notation which will be used in our theorems.

Definition 2.1 (*see [2,3,13,14]*). The Riemann–Liouville fractional integral of order $\alpha > 0$ of a Lebesgue-measurable function $f: R^+ \to R$ is defined by

$$I_a^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}f(s)ds,$$

when $a = 0$ we write $I_a^{\alpha}f(t) = I^{\alpha}f(t)$.

And we have, for $\alpha, \beta \in R^+$,

(r_1) $I_a^{\alpha}: L_1 \longrightarrow L_1$,
(r_2) $f(t) \in L_1, \quad I_a^{\alpha}\, I_a^{\beta}f(t) = I_a^{\alpha+\beta}f(t)$.

Definition 2.2 (*see [2,3,13,14]*). The Riemann–Liouville fractional derivative of order $\alpha \in (0,1]$ of a Lebesgue-measurable function $f: R^+ \to R$ is defined by

$$D^{\alpha}f(t) = \frac{d}{dt}I^{1-\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}f(s)ds.$$

Theorem 2.1 (*Schauder fixed point Theorem*). *Let S be a nonempty, closed , convex and bounded subset of the Banach space X and let $Q: S \to S$ be a continuous and compact operator. Then the operator equation $Qu = u$ has at least one fixed-point in S.*

Theorem 2.2 (*Kolmogorov compactness criterion [20]*). *Let $\Omega \subseteq L^p(0,T), 1 \leqslant p < \infty$. If*

(i) *Ω is bounded in $L^p(0,T)$ and*
(ii) *$u_h \to u$ as $h \to 0$ uniformly with respect to $u \in \Omega$, then Ω is relatively compact in $L^p(0,T)$ where*

$$u_h(t) = \frac{1}{h}\int_t^{t+h} u(s)ds.$$

Definition 2.3. A function $f: I \times R \to R$ is called Carathéodory function if:

(i) $t \to f(t,u)$ is measurable for all $u \in R$, and,
(ii) $u \to f(t,u)$ is continuous for all $t \in I$.
(iii) There exists a Lebesgue function $h(t)$ on I, and

3. Integral equation representation

We investigate in our paper the Cauchy problem for the nonlinear fractional differential equation with the nonlocal condition with the following assumptions.

(h_1) The function $f: (0,T] \times R \to R$ is Carathéodory function.
(h_2) $p(t) > 0$ for all $t \in I$ and is continuous with $\inf_{(0,T]}|p(t)| = p$.
(h_3) $\sum_{j=1}^{m} \frac{a_j}{p(\tau_j)\tau_j^{1-\alpha}} \neq 1$.

Lemma 3.1. *The solution of the nonlocal problem (1) and (2) can be expressed by the fractional-order integral equation*

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