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ORIGINAL ARTICLE

Subordination and superordination preserving properties for a family of integral operators involving the Noor integral operator

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Abstract In the present paper, we introduce a family of integral operators $\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)$ associate with the Noor integral operator in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$, which is defined by the convolution $[f_{p,\delta}^{\mu}(a,b,c)(z)]^{(-1)} * f(z)$, where

$$f_{p,\delta}^{\mu}(a,b,c)(z)=(1-\mu+\delta)z^p{}_2F_1(a,b;c;z)+(\mu-\delta)z[z^p{}_2F_1(a,b;c;z)]'+\mu\delta z^2[z^p{}_2F_1(a,b;c;z)]''$$
$$(p\in\mathbb{N}=\{1,2,\cdots\};\mu,\delta\geqslant 0;z\in\mathbb{U}).$$

By using the operator $\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)$, we investigate some subordination and superordination preserving properties for certain classes of analytic and multivalent functions in $\mathbb{U}$. Various sandwich-type results for these multivalent functions are also obtained.

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1. Introduction

Let $\mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. For $\mathfrak{a}\in\mathbb{C}$ and $n\in\mathbb{N}=\{1,2,\cdots\}$, let

$$\mathcal{H}[\mathfrak{a},n]=\{f\in\mathcal{H}(\mathbb{U}):f(z)=\mathfrak{a}+a_nz^n+a_{n+1}z^{n+1}+\cdots\}.$$

Let f and g be two members of $\mathcal{H}(\mathbb{U})$. The function f is said to be subordinate to g, or g is said to be superordinate to f, if there exists a Schwarz function ω, analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1 (z\in\mathbb{U})$, such that $f(z)=g(\omega(z)) (z\in\mathbb{U})$. In such a case, we write $f\prec g$ or

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$f(z) \prec g(z) (z \in \mathbb{U})$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have (see [1,2]):

$f \prec g\ (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Definition 1.1 (see [1]). Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If $\mathfrak{p}$ is analytic in $\mathbb{U}$ and satisfies the following differential subordination

$$\phi(\mathfrak{p}(z), z\mathfrak{p}'(z)) \prec h(z)\ (z \in \mathbb{U}), \tag{1.1}$$

then $\mathfrak{p}$ is called a solution of the differential subordination (1.1). The univalent function $\mathfrak{q}$ is called a dominant of the solutions of the differential subordination (1.1), if $\mathfrak{p} \prec \mathfrak{q}$ for all $\mathfrak{p}$ satisfying (1.1). A dominant $\tilde{\mathfrak{q}}$ that satisfies $\tilde{\mathfrak{q}} \prec \mathfrak{q}$ for all dominants $\mathfrak{q}$ of (1.1) is said to be the best dominant.

Definition 1.2 (see [3]). Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If $\mathfrak{p}$ and $\varphi(\mathfrak{p}(z), z\mathfrak{p}'(z))$ are univalent in $\mathbb{U}$ and satisfy the following differential superordination

$$h(z) \prec \varphi(\mathfrak{p}(z), z\mathfrak{p}'(z))\ (z \in \mathbb{U}), \tag{1.2}$$

then $\mathfrak{p}$ is called a solution of the differential superordination (1.2). An analytic function $\mathfrak{q}$ is called a subordination of the solutions of the differential superordination (1.2), if $\mathfrak{q} \prec \mathfrak{p}$ for all $\mathfrak{p}$ satisfying (1.2). A univalent subordination $\tilde{\mathfrak{q}}$ that satisfies $\mathfrak{q} \prec \tilde{\mathfrak{q}}$ for all subordinations $\mathfrak{q}$ of (1.2) is said to be the best subordination.

Definition 1.3 (see [3]). We denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\overline{\mathbb{U}} \setminus E(f)$, where

$$E(f) = \left\{\xi : \xi \in \partial\mathbb{U} \text{ and } \lim_{z\to\xi} f(z) = \infty\right\},$$

and are such that $f'(\xi) \neq 0 (\xi \in \partial\mathbb{U} \setminus E(f))$.

Let $\mathcal{A}_n(p)$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k}\ (p, n \in \mathbb{N}; z \in \mathbb{U}),$$

and let $\mathcal{A}_1(p) = \mathcal{A}(p)$.

For $f \in \mathcal{A}(p)$, we denote by $\mathcal{D}^{n+p-1} : \mathcal{A}(p) \to \mathcal{A}(p)$ the operator defined by

$$\mathcal{D}^{n+p-1} f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z)\ (n > -p)$$

or, equivalently, by

$$\mathcal{D}^{n+p-1} f(z) = \frac{z^p (z^{n-1} f(z))^{(n+p-1)}}{(n+p-1)!},$$

where n is any integer greater than $-p$ and the symbol $(*)$ stands for the Hadamard product (or convolution). The operator $\mathcal{D}^{n+p-1}$ with $p = 1$ was introduced by Ruscheweyh [4], and $\mathcal{D}^{n+p-1}$ was introduced by Goel and Sohi [5]. The operator $\mathcal{D}^{n+p-1}$ is called as the Ruscheweyh derivative of $(n + p - 1)$th order.

Recently, analogous to $\mathcal{D}^{n+p-1}$, Liu and Noor [6] introduced an integral operator $\mathcal{I}_{n,p} : \mathcal{A}(p) \to \mathcal{A}(p)$ as below.

Let $f_{n,p}(z) = z^p/(1-z)^{n+p} (n > -p)$, and let $f_{n,p}^{(\dagger)}(z)$ be defined such that

$$f_{n,p}(z) * f_{n,p}^{(\dagger)}(z) = \frac{z^p}{(1-z)^{p+1}}.$$

Then

$$\mathcal{I}_{n,p} f(z) = f_{n,p}^{(\dagger)}(z) * f(z) = \left(\frac{z^p}{(1-z)^{n+p}}\right)^{(\dagger)} * f(z)\ (f \in \mathcal{A}(p)). \tag{1.3}$$

We note that $\mathcal{I}_{0,p} f(z) = zf'(z)/p$ and $\mathcal{I}_{1,p} f(z) = f(z)$. Also, the operator $\mathcal{I}_{n,p}$ defined by (1.3) is called the Noor integral operator $(n + p - 1)$-th order [6]. For $p = 1$, the operator $\mathcal{I}_{n,1} \equiv \mathcal{I}_n$ was introduced by Noor [7] and Noor and Noor [8], which is an important operator in defining several classes of analytic functions. In recent years, it has been shown that Noor integral operator has fundamental and significant applications in analytic function theory. For the properties and applications of the Noor integral operator, see, for example, [9–13].

For real or complex numbers a, b, c other than $0, -1, -2, \cdots$, the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{1.4}$$

where $(v)_k$ denotes the Pochhammer symbol defined, in terms of Gamma function, by

$$(v)_k = \frac{\Gamma(v+k)}{\Gamma(v)} = \begin{cases} 1 & (k = 0), \\ v(v+1)\cdots(v+k-1) & (k \in \mathbb{N}). \end{cases}$$

Since the series in (1.4) converges absolutely for all $z \in \mathbb{U}$, so that it represents an analytic function in $\mathbb{U}$.

We now introduce a function $f_{p,\delta}^{\mu}(a, b, c)(z)$ defined by

$$\begin{aligned} f_{p,\delta}^{\mu}(a, b, c)(z) = {} & (1 - \mu + \delta) z^p {}_2F_1(a, b; c; z) \\ & + (\mu - \delta) z [z^p {}_2F_1(a, b; c; z)]' \\ & + \mu\delta z^2 [z^p {}_2F_1(a, b; c; z)]'' (p \in \mathbb{N}; \mu, \delta \geqslant 0; z \in \mathbb{U}). \end{aligned}$$

In its special case when $p = 1$ and $\delta = 0$, we obtain $f_{1,0}^{\mu}(a, b, c)(z) = f_{\mu}(a, b, c)(z)$ studied by Shukla and Shukla [14].

On the other hand, we define a function $[f_{p,\delta}^{\mu}(a, b, c)(z)]^{(-1)}$ by means of Hadamard product (or convolution):

$$f_{p,\delta}^{\mu}(a, b, c)(z) * \left[f_{p,\delta}^{\mu}(a, b, c)(z)\right]^{(-1)} = \frac{z^p}{(1-z)^{\lambda+p}} \quad (\mu, \delta \geqslant 0; \lambda > -p),$$

which leads us to the following family of linear operators

$$\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c) f(z) = [f_{p,\delta}^{\mu}(a, b, c)(z)]^{(-1)} * f(z), \tag{1.5}$$

where a, b, c are real numbers other than $0, -1, -2, \cdots$, and $f \in \mathcal{A}_n(p)$.

We observe that the operator $\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)$ generalizes several previously studied familiar operators, and we will show some of the interesting particular cases as follows.

(i) $\mathcal{I}_{1,1,0}^{\lambda,\mu}(a, b, c) = \mathcal{I}_{\mu}^{\lambda}(a, b, c)$, where $\mathcal{I}_{\mu}^{\lambda}(a, b, c)$ is the Srivastava et al. operator [15];
(ii) $\mathcal{I}_{p,n,0}^{\lambda,0}(a, b, c) = \mathcal{I}_{p,n}^{\lambda}(a, b, c)$, where the operator $\mathcal{I}_{p,n}^{\lambda}(a, b, c)$ was introduced by Fu and Liu [16];