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On a discretization process of fractional-order Logistic differential equation

Z.F. El Raheem, S.M. Salman *

Faculty of Education, Alexandria University, Alexandria, Egypt

Abstract   In this work we introduce a discretization process to discretize fractional-order differential equations. First of all, we consider the fractional-order Logistic differential equation then, we consider the corresponding fractional-order Logistic differential equation with piecewise constant arguments and we apply the proposed discretization on it. The stability of the fixed points of the resultant dynamical system and the Lyapunov exponent are investigated. Finally, we study some dynamic behavior of the resultant systems such as bifurcation and chaos.

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1. Introduction

Chaotic systems have been a focal point of renewed interest for many researchers in the past few decades. Such nonlinear systems can occur in various natural and man-made systems, and are known to have great sensitivity to initial conditions. In recent years differential equations with fractional-order have attracted many researchers because of their applications in many areas of science and engineering. Analytical and numerical techniques have been implemented to study such equations. The fractional calculus has allowed the operations of integra-

tion and differentiation to be applied upon any fractional-order. For the existence of solutions for fractional differential equations, one can see [1,2].

About the development of existence theorems for fractional functional differential equations, many contributions existed and can be referred to [3–5]. Many applications of fractional calculus amounts to replace the time derivative in a given evolution equation by a derivative of fractional-order.

Recalling the basic definitions (Caputo) and properties of fractional-order differentiation and itegration

Definition 1. The fractional integral of order $\beta \in \mathbb{R}^+$ of the function $f(t)$, $t > 0$ is defined by

$$I^\beta f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)ds,$$

and the fractional derivative of order $\alpha \in (n-1, n)$ of $f(t)$, $t > 0$ is defined by

* Corresponding author. Tel.: +20 1091103980.
E-mail addresses: zaki55@hotmail.com (Z.F. El Raheem), samastars9 @gmail.com (S.M. Salman).
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$$D^\alpha f(t) = I^{n-\alpha} D^n f(t), \quad D = \frac{d}{dt}.$$

To solve fractional-order differential equations there are two famous methods: frequency domain methods [6] and time domain methods [7]. In recent years it has been shown that the second method is more effective because the first method is not always reliable in detecting chaos [8,9].

Often it is not desirable to solve a differential equation analytically, and one turns to numerical or computational methods.

In [10], a numerical method for nonlinear fractional-order differential equations with constant or time-varying delay was devised. It should be noticed that the fractional differential equations tend to lower the dimensionality of the differential equations in question, however, introducing delay in differential equations makes it infinite dimensional. So, even a single ordinary differential equation with delay could display chaos.

On the other hand, some examples of dynamical systems generated by piecewise constant arguments have been studied in [11–14]. Here we propose a discretization process to obtain the discrete version of the system under study. Mean while, we apply discretization process to discretize the fractional-order Logistic differential equation.

A lot of differential equations with Caputo fractional derivative were simulated by the Predictor-Corrector scheme, such as the fractional Chua system, the fractional Chen system, and Lorenz system. We should note that Predictor-Corrector method is an approximation for the fractional-order integration, however, our approach is an approximation for the right hand side. For applications of fractional-order differential equations one can see [15–17], [21], and [23–27].

2. Discretization process

Consider the fractional-order Logistic differential equation given by

$$D^\alpha x(t) = \rho x(t)(1 - x(t)), \quad t > 0, \tag{2.1}$$

with the initial condition $x(0) = x_o$.

The main purpose of this section is to introduce a discretization process to discretize the counterpart of (2.1) with piecewise constant arguments

$$D^\alpha x(t) = \rho x\left(\left[\frac{t}{r}\right]r\right)\left(1 - x\left(\left[\frac{t}{r}\right]r\right)\right), \tag{2.2}$$

with the initial condition $x(0) = x_o$.

We proceed like the step method mentioned in [20] and [22]. The steps of the discretization process is as follows

(1) Let $t \in [0, r)$, then $\frac{t}{r} \in [0, 1)$. So, we get

$$D^\alpha x(t) = \rho x_o(1 - x_o), \quad t \in [0, r),$$

and the solution of (2.2) is given by

$$x_1(t) = x_o + I^\alpha \rho x_o(1 - x_o) = x_o + \rho x_o(1 - x_o) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds$$

$$= x_o + \rho x_o(1 - x_o) \frac{t^\alpha}{\Gamma(1 + \alpha)}$$

(2) Let $t \in [r, 2r)$, then $\frac{t}{r} \in [1, 2)$. So, we get

$$D^\alpha x(t) = \rho x_1(1 - x_1), \quad t \in [r, 2r),$$

and the solution of (2.2) is given by

$$x_2(t) = x_1(r) + I_r^\alpha \rho x_1(1 - x_1)$$

$$= x_1(r) + \rho x_1(1 - x_1) \int_r^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds$$

$$= x_1(r) + \rho x_1(r)(1 - x_1(r)) \frac{(t-r)^\alpha}{\Gamma(1 + \alpha)}$$

Repeating the process we can easily deduce that the solution of (2.2) is given by

$$x_{n+1}(t) = x_n(nr) + \frac{(t-nr)^\alpha}{\Gamma(1 + \alpha)} \rho x_n(nr)(1 - x_n(nr)), \quad t$$

$$\in [nr, (n+1)r).$$

Let $t \to (n + 1)r$, we obtain the discretization

$$x_{n+1}((n+1)r) = x_n(nr) + \frac{r^\alpha}{\Gamma(1 + \alpha)} \rho x_n(nr)(1 - x_n(nr)),$$

That is

$$x_{n+1} = x_n + \frac{r^\alpha}{\Gamma(1 + \alpha)} \rho x_n(1 - x_n). \tag{2.3}$$

On a similar manner, consider the corresponding equation of (2.1) with piecewise constant arguments

$$D^\alpha x(t) = \rho x\left(\left[\frac{t}{r}\right]r\right)\left(1 - x\left(\left[\frac{t-r}{r}\right]r\right)\right), \tag{2.4}$$

with the initial condition $x(0) = x_o$. So, we obtain the second order discretization

$$x_{n+1} = x_n + \frac{r^\alpha}{\Gamma(1 + \alpha)} \rho x_n(1 - x_{n-1}). \tag{2.5}$$

3. Fixed points and their asymptotic stability

Now we study the stability of the fixed points of the Eq. (2.3) which has two fixed points namely, 0 and 1 given by solving the equation

$$x = x + \frac{r^\alpha}{\Gamma(1 + \alpha)} \rho x(1 - x).$$

To study the stability of these fixed points we relay on the following theorem

Theorem 1 [18]. *Let f be a smooth map on $\mathbb{R}$, and assume that x_0 is a fixed point of f.*

1. If $-f'(x_0)- < 1$, then x_0 is stable.
2. If $-f'(x_0)- > 1$, then x_0 is unstable.

In case of the first fixed point '0', it is stable if $\left|1 + \frac{r^\alpha}{\Gamma(1+\alpha)}\rho\right| < 1$ which is impossible. That is the origin is unstable. For the second fixed point '1', it is stable if

$$0 < \rho < \frac{2(\Gamma(1 + \alpha))}{r^\alpha}. \tag{3.1}$$

On the other hand, to study the stability of the fixed points of Eq. (2.5) we first split it into two equations as follows