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ORIGINAL ARTICLE

On some Toeplitz matrices and their inversions

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Abstract In this article, using the difference operator $B(a[m])$, we introduce a lower triangular Toeplitz matrix T which includes several difference matrices such as $\Delta^{(1)}$, $\Delta^{(m)}$, $B(r,s)$, $B(r,s,t)$, and $B(\tilde{r},\tilde{s},\tilde{t},\tilde{u})$ in different special cases. For any $x \in w$ and $m \in \mathbb{N}_0 = \{0,1,2,\ldots\}$, the difference operator $B(a[m])$ is defined by $(B(a[m])x)_k = a_k(0)x_k + a_{k-1}(1)x_{k-1} + a_{k-2}(2)x_{k-2} + \cdots + a_{k-m}(m)x_{k-m}$, $(k \in \mathbb{N}_0)$ where $a[m] = \{a(0), a(1), \ldots, a(m)\}$ and $a(i) = (a_k(i))$ for $0 \leqslant i \leqslant m$ are convergent sequences of real numbers. We use the convention that any term with negative subscript is equal to zero. The main results of this article relate to the determination and applications of the inverse of the Toeplitz matrix T.

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1. Introduction

Let w be the space all real valued sequences. We write $a[m]$ for any convergent sequence $a(i) = (a_k(i))$ of real numbers satisfying $a(i) \neq a(j)$, where $m \in \mathbb{N}_0$ and $0 \leqslant i, j \leqslant m$. Let $x = (x_k)$ be any sequence in w, then we define the generalized difference operator $B(a[m])$ as follows:

$$(B(a[m])x)_k = a_k(0)x_k + a_{k-1}(1)x_{k-1} + a_{k-2}(2)x_{k-2} + \cdots + a_{k-m}(m)x_{k-m}, \quad (k \in \mathbb{N}_0). \tag{1.1}$$

We assume throughout that any term with negative subscript is zero. It is natural that the difference operator given in Eq. (1.1), can be expressed as a lower triangular Toeplitz matrix $T = (b_{nk})$, where

$$(b_{nk}) = \begin{pmatrix} a_0(0) & 0 & 0 & \ldots & 0 & 0 & \ldots \\ a_0(1) & a_1(0) & 0 & \ldots & 0 & 0 & \ldots \\ a_0(2) & a_1(1) & a_2(0) & \ldots & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_0(m) & a_1(m-1) & a_2(m-2) & \ldots & a_m(0) & 0 & \ldots \\ 0 & a_1(m) & a_2(m-1) & \ldots & a_m(1) & a_{m+1}(0) & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In particular, the difference operator $B(a[m])$ has the following generalizations:

(i) For $a(0) = e = (1, 1, 1, \ldots)$, $a(1) = -e$ and $a(i) = \theta = (0, 0, 0, \ldots)$ for $2 \leqslant i \leqslant m$, the difference matrix $B(a(m))$ reduces to $\Delta^{(1)}$ studied by Kızmaz [1].

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(ii) For $a(0) = e$, $a(1) = -2e$, $a(2) = e$ and $a(i) = \theta$ for $3 \leqslant i \leqslant m$, the difference matrix $B(a(m))$ reduces to Δ^2 studied by Dutta and Baliarsingh [2].
(iii) For $a(0) = re$, $a(1) = se$, $0 \neq r$, $s \in \mathbb{R}$ and $a(i) = \theta$ for $2 \leqslant i \leqslant m$, the difference matrix $B(a(m))$ reduces to $B(r, s)$ studied by Altay and Başar [3].
(iv) For $a(0) = re$, $a(1) = se$, $a(2) = te$, $0 \neq r, s, t \in \mathbb{R}$ and $a(i) = \theta$ for $3 \leqslant i \leqslant m$, the difference matrix $B(a(m))$ reduces to $B(r, s, t)$ studied by Furkan et al. [4].
(v) For $a(i) = \binom{m}{i}$ for $0 \leqslant i \leqslant m$ and $m = r$, the difference matrix $B(a(m))$ reduces to Δ^r studied by Dutta and Baliarsingh [5].
(vi) For $a(0) = re$, $a(1) = se$, $a(2) = te$, $a(3) = ue$, $0 \neq r, s, t, u \in \mathbb{R}$ and $a(i) = \theta$ for $4 \leqslant i \leqslant m$, the difference matrix $B(a(m))$ reduces to $B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})$ studied by Baliarsingh and Dutta [6].

For last several decades, many new theories and fundamental results have been introduced and studied by different authors contributing to the development of sequence spaces. Amongst all, one of the most interesting idea is the study of sequence spaces by using difference matrices. For example: Kızmaz [1] introduced the difference matrix Δ and studied the sequence spaces $X(\Delta)$, for $X = \ell_\infty$, c, c_0, Et and Çolak [7] generalized these results by introducing the generalized difference matrix $\Delta^m, (m \in \mathbb{N}_0)$ and Baliarsingh [8] studied the difference sequence spaces $\lambda(\Gamma, \Delta^\alpha, u)$ for $\lambda \in \{\ell_\infty, c_0, c\}$ by introducing the difference matrix $\Delta^\alpha, (\alpha \in \mathbb{R})$. The difference matrices $B(r, s)$, $B(r, s, t)$ and $B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})$ have been introduced and studied by Altay and Başar [3], Furkan et al. [4] and Baliarsingh and Dutta [6], respectively. Recently, using difference matrices, various sequence spaces have been defined and different results concerning their topological properties, matrix transformations, spectral properties and many more (see [8–28]) have been established. The main objective of this work is to define a generalized difference operator and unify most of the difference matrices defined earlier and establish certain results concerning its inverse.

2. Main results

The most general and effective application of the difference matrix $a[m]$ is to redefine some triangles and find their inverses. In the present section, we redefine some well known lower triangular matrices such as generalized Fibonacci, Pascal and weighted mean factorable difference matrices, and we obtain some results related to the linearity, boundedness and inverse of the difference matrix $B(a[m])$.

Let F_n, $(n \in \mathbb{N}_0)$ be the nth Fibonacci number which satisfies the recurrence relation $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$, $F_1 = 1$. Then for any $s, t \in \mathbb{R}$, we define generalized lower triangular Fibonacci matrix $F(r, s)$ as follows:

$$(F(r,s))_{nk} = \begin{cases} r, & (k = n) \\ r + s, & (k = n-1) \\ F_{n-1}r + F_{n-2}s, & (0 \leqslant k \leqslant n-2) \\ 0, & (k > n) \end{cases}, \quad (n, k \in \mathbb{N}_0).$$

Clearly, for $r = 0$, $s = 1$, the matrix $F(r, s)$ reduces to the usual Fibonacci matrix F studied in [28,29]. The lower triangular Pascal matrix $P = (p_{nk})$ is defined by

$$p_{nk} = \begin{cases} \binom{n}{n-k}, & (0 \leqslant k \leqslant n) \\ 0, & (k > n) \end{cases}, \quad (n, k \in \mathbb{N}_0).$$

The well known weighted mean factorable difference matrix $G(u, v; \Delta) = \left(g^\Delta_{nk}\right)$ is defined as follows:

$$g^\Delta_{nk} = \begin{cases} u_n v_n, & (k = n) \\ u_n(v_k - v_{k+1}), & (0 \leqslant k \leqslant n-1) \\ 0, & (k > n). \end{cases}, \quad (n, k \in \mathbb{N}_0),$$

where we write $\mathcal{U}$ for the set of all sequences $u = (u_n)$ such that $u_n \neq 0$ for all $n \in \mathbb{N}_0$, and $u, v \in \mathcal{U}$. Now, we state some important theorems.

Theorem 1. *The difference operator $B(a[m])$: $w \to w$ is a linear operator and satisfying*

$$\|B(a[m])\| = \sup_k (|a_k(0)| + |a_k(1)| + \cdots + |a_k(m)|).$$

Proof. The proof is trivial, so we omit it. □

Theorem 2. *If $a_k(0) \neq 0$ for all $k \in \mathbb{N}_0$, then the inverse of the difference operator $B(a[m])$ is given by a lower triangular Toeplitz matrix $C = (c_{nk})$ as follows:*

$$c_{nk} = \begin{cases} \frac{1}{a_n(0)}, & (k = n), \\ \frac{\frac{(-1)^{n-k}}{n}}{\prod_{j=k} a_j(0)} D^{(k)}_{n-k}(a[m]), & (0 \leqslant k \leqslant n-1), \\ 0, & (k > n), \end{cases} \quad (n, k \in \mathbb{N}_0),$$

where

$$D^{(k)}_n(a[m]) = \begin{vmatrix} a_k(1) & a_{k+1}(0) & 0 & \dots & 0 & 0 & \dots & 0 \\ a_k(2) & a_{k+1}(1) & a_{k+2}(0) & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_k(m) & a_{k+1}(m-1) & a_{k+2}(m-2) & \dots & a_{m-1}(1) & a_m(0) & \dots & 0 \\ 0 & a_{k+1}(m) & a_{k+2}(m-1) & \dots & a_{m-1}(2) & a_m(1) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n-m}(m) & \dots & \dots & \dots & a_{n-1}(1) \end{vmatrix}.$$

Proof. The proof is clear from the following examples. □

Examples

(i) The inverse of the difference matrix $\Delta^{(1)}$ is

$$\left(\left(\Delta^{(1)}\right)^{-1}\right)_{nk} = \begin{cases} 1, & (0 \leqslant k \leqslant n) \\ 0, & (k > n) \end{cases}, \quad (n, k \in \mathbb{N}_0).$$

This follows from the fact that $a(0) = e$, $a(1) = -e$, $a(2) = a(3) \ldots = a(m) = \theta$ and $D^{(0)}_n(\Delta^{(1)}) = (-1)^n$.

(ii) The inverse of the difference matrix $\Delta^{(m)}, m \in \mathbb{N}_0$ is

$$\left(\left(\Delta^{(m)}\right)^{-1}\right)_{nk} = \begin{cases} 1, & (k = n) \\ \frac{m(m+1)\ldots(m+n-k)}{(n-k)!}, & (0 \leqslant k \leqslant n-1) \\ 0, & (k > n) \end{cases}, \quad (n, k \in \mathbb{N}_0).$$