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ORIGINAL ARTICLE

Convergence theorems for three finite families of multivalued nonexpansive mappings

Rashwan Ahmed *, **Saeed Altwqi**

Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt

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Abstract In this paper, we obtain weak and strong convergence theorems of an iterative sequences associated with three finite families of multivalued nonexpansive mappings under some conditions in a uniformly convex real Banach space. Our results extend and improve several known results.

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1. Introduction and preliminaries

Let E be a Banach space with dim $E \geqslant 2$, the modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| : \|x\| = 1, \|y\| = 1, \|x-y\| = \epsilon\right\}.$$

E is uniformly convex if and only if with $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$.

A subset K is called proximinal if for each $x \in E$, there exists an element $k \in K$ such that

$$d(x,k) = \inf\{\|x-y\| : y \in K\} = d(x,K).$$

It is known that a weakly compact convex subsets of a Banach space and closed convex subsets of a uniformly convex Banach space are proximinal. We shall denote the family of nonempty bounded proximinal subsets of K by $P(K)$, $C(K)$ the family of nonempty compact subsets of K, and $CB(K)$ be the class of all nonempty bounded and closed subsets of K, Consistent with [1].

Let H be a Hausdorff metric induced by the metric d of K, given by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\},$$

for every $A, B \in CB(K)$. It is obvious that $P(K) \in CB(K)$.

A multivalued mapping $T : K \to P(K)$ is said to be a contraction if there exists a constant $k \in [0,1)$ such that for any $x, y \in K$,

$$H(Tx,Ty) \leqslant k\|x-y\|,$$

and T is said to be nonexpansive if

$$H(Tx,Ty) \leqslant \|x-y\|,$$

* Corresponding author. Tel.: +20 1094352604.
E-mail addresses: rr_rashwan54@yahoo.com (R. Ahmed), Saeedaltwqi@yahoo.com (S. Altwqi).
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for all $x, y \in K$. A point $x \in K$ is called a fixed point of T if $x \in Tx$. Throughout the paper $\mathbb{N}$ denotes the set of all natural numbers and $F(T)$ the set of fixed points of T.

Let us recall the following definitions.

Definition 1.1 [2]. A Banach space E is said to satisfy Opial's condition if for any sequence $\{x_n\}$ in $E, x_n \rightharpoonup x$ ($\rightharpoonup$ denotes weak convergence) implies that $\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|$ for all $y \in E$ with $y \neq x$. Examples of Banach spaces satisfying this condition are Hilbert spaces and all $1 < p < \infty$.

Definition 1.2 [3]. The mapping $T : K \to K$ where K a subset of E, are said to satisfy condition (A) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0,\infty)$ such that either $d(x, Tx) \geqslant f(d(x, F(T))$ for all $x \in K$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$

The following is the multivalued version of condition (Á);

Definition 1.3. The three finite families of multivalued nonexpansive mappings $T_i, S_i, R_i : K \to CB(K), (i = 1, 2, 3, \ldots, k)$, where K a subset of E, are said to satisfy condition $(\grave{A})$ if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0,\infty)$ such that $d(x, T_i x) \geqslant f(d(x,\mathbf{F})$ or $d(x, S_i x) \geqslant f(d(x,\mathbf{F})$ or $d(x, R_i x) \geqslant f(d(x,\mathbf{F})$ for all $x \in K$, where $\mathbf{F} = \left(\bigcap_{i=1}^{k} F(T_i)\right) \cap \left(\bigcap_{i=1}^{k} F(S_i)\right) \cap \left(\bigcap_{i=1}^{k} F(R_i)\right)$, the set of all common fixed points of the mappings T_i, S_i and R_i.

Definition 1.4 [4]. A map $T : K \to CB(K)$, is called hemicompact if, for any sequence $\{x_n\}$ in E such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ such that $x_{n_r} \to p \in K$. We note that if K is compact, then every multivalued mapping $T : K \to CB(K)$ is hemicompact.

Next we state the following useful lemma.

Lemma 1.1 [5]. *Let E be a uniformly convex Banach space, $r > 0$ a positive number and let $B_r(0) = \{x \in E : \|x\| \leqslant r\}$. Then, for any given sequence $\{x_i\} \subset B_r(0)$ and for any given sequence $\lambda_i \in [0,1]$ with $\sum_{i=0}^{k}\lambda_i = 1$, there exists a continuous strictly increasing and convex function $\varphi : [0, 2r) \to \mathbb{R}, \varphi(0) = 0$ such that for any positive integers m, j with $m < j$, the following inequality holds:*

$$\left\|\sum_{i=1}^{k}\lambda_i x_i\right\|^2 \leqslant \sum_{i=1}^{k}\lambda_i\|x_i\|^2 - \lambda_m\lambda_j\varphi(\|x_m - x_j\|).$$

The study of fixed points for multivalued contractions and nonexpansive mappings using the Hausdorff metric was initiated by Markin [6] and Nadler [1]. Later, an interesting and rich fixed point theory for such maps was developed which has applications in control theory, convex optimization, differential inclusion and economics.

The theory of multivalued nonexpansive mappings is harder than the corresponding theory of single valued nonexpansive mappings. Different iterative processes have been used to approximate the fixed points of multivalued nonexpansive mappings. In particular in 2005, Sastry and Babu [7] proved the convergence of Mann and Ishikawa iteration process for multivalued mapping T with a fixed point p converge to a fixed point q of T under certain conditions. They also claimed that the fixed point q may be different from p. Under some conditions Panyanak [8] extended result of Sastry and Babu to uniformly convex Banach spaces. Song and Wang [9] noted that there was a gap in the proof of the main result in [8]. They further revised the gap and also gave the affirmative answer to Panyanak's open question.

Abbas et al. [10] established weak and strong convergence theorems of two multivalued nonexpansive mappings in a uniformly convex real Banach space by one-step iterative process to approximate common fixed points under some basic boundary conditions. Rashwan and Altwqi [11] introduced a new one-step iterative process to approximate the common fixed points of three multivalued nonexpansive mappings.

Recently Eslamian and Abkar [12] introduced a new one-step iterative process for approximate the common fixed points of finitely many multivalued mappings satisfying some conditions. They proved some weak and strong convergence theorems for such iterative process in uniformly convex Banach spaces as follows. Let E be a Banach space, K be a nonempty convex subset of E and $T_i : K \to CB(K)$ $(i = 1, 2, \ldots, m)$ be finitely many given mappings. Then, for $x_0 \in K$ and they defined:

$$x_{n+1} = a_{n,0}x_n + \sum_{i=1}^{m} a_{n,i}z_{n,i}, \quad n \in \mathbb{N}, \tag{1.1}$$

where $z_{n,i} \in T_i(x_n)$ and $\{a_{n,i}\}$ are sequences of numbers in $[0,1]$ such that for every natural number $n \in \mathbb{N}$ and $\sum_{i=0}^{m} a_{n,i} = 1$.

We now introduce the following iteration scheme which attend (1.1). Let E be Banach space, K be a nonempty closed convex subset of E and let $T_i, S_i, R_i : K \to CB(K)$, $(i = 1, 2, \ldots, k)$ be three finite families of multivalued mappings. Then for $x_0 \in K$, define the sequences $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ by:

$$\begin{aligned} x_{n+1} &= \alpha_{n,0}x_n + \sum_{i=1}^{k}\alpha_{n,i}u_{n,i}, \\ y_n &= \beta_{n,0}x_n + \sum_{i=1}^{k}\beta_{n,i}v_{n,i}, \quad n \in \mathbb{N} \\ z_n &= \gamma_{n,0}x_n + \sum_{i=1}^{k}\gamma_{n,i}w_{n,i}, \end{aligned} \tag{1.2}$$

where $u_{n,i} \in T_i y_n, v_{n,i} \in S_i z_n, w_{n,i} \in R_i x_n$ and $\{\alpha_{n,i}\}, \{\beta_{n,i}\}$ and $\{\gamma_{n,i}\}$ are sequence of numbers in $[0,1]$ satisfying $\sum_{i=0}^{k}\alpha_{n,i} = \sum_{i=0}^{k}\beta_{n,i} = \sum_{i=0}^{k}\gamma_{n,i} = 1$.

Remark 1.1

1. If $\beta_{n,0} = 1$ $\gamma_{n,0} = 1$ and $\sum_{i=1}^{k}\beta_{n,i} = \sum_{i=1}^{k}\gamma_{n,i} \equiv 0$. The iterative scheme (1.2) reduce to iterative scheme defined by (1.1).
2. If $\sum_{i=2}^{k}\alpha_{n,i} = \sum_{i=2}^{k}\beta_{n,i} = \sum_{i=2}^{k}\gamma_{n,i} \equiv 0$. The iterative scheme (1.2) reduce to Noor iterative scheme defined by

$$\begin{aligned} x_{n+1} &= \alpha_{n,0}x_n + \alpha_{n,1}u_{n,1}, \\ y_n &= \beta_{n,0}x_n + \beta_{n,1}v_{n,1}, \quad n \in \mathbb{N} \\ z_n &= \gamma_{n,0}x_n + \gamma_{n,1}w_{n,1}, \end{aligned} \tag{1.3}$$

where $u_{n,1} \in T_1 y_n$, $v_{n,1} \in S_1 z_n$ and $w_{n,1} \in R_1 x_n$.