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ORIGINAL ARTICLE

A strong convergence theorem for a modified Krasnoselskii iteration method and its application to seepage theory in Hilbert spaces

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Abstract Inspired by the modified iteration method devised by He and Zhu [1], the purpose of this paper is to present a modified Krasnoselskii iteration via boundary method. A strong convergence theorem of this iteration for finding minimum norm solution of nonlinear equation of the form $S_{h(x)}(x) = 0$, where $S_{h(x)}$ is a nonlinear mapping of C into itself and h is a function of C into $[0, 1]$ is then proved in Hilbert spaces. In the same vein, an application to the stationary problem of seepage theory is also presented. The results of this paper are extensions and improvements of some earlier theorems of Saddeek et al. [2].

MATHEMATICS SUBJECT CLASSIFICATION: 47H06; 47H10; 49M05; 90C25

1. Introduction

Fixed point theory is an interesting topic with multiple applications in various branches of mathematics. For example in fluid mechanics, the development of iteration methods for finding fixed points of nonlinear self-mappings has historically been an important enterprise.

The Krasnoselskii iteration method (KIM) is one of the most salient examples of these methods.

Let X be a real Hilbert space and let C be a nonempty, closed and convex subset of X. Let $T : C \to C$ be a self-mapping.

Then the KIM (see, for example, [3]) generates, for any $x_0 \in C$, a sequence $\{x_n\}$ in C by

$$x_{n+1} = (1 - \tau)x_n + \tau T x_n, \quad n \geqslant 0, \tag{1}$$

where $\tau \in [0, 1]$.

It should be noted that for $\tau = 1$, the KIM reduces to the Picard iteration (successive iteration) method (see, for example, [4]) that is $x_{n+1} = Tx_n, n \geqslant 0$.

The KIM has been studied extensively by many authors (see, for example, [5–11]).

In a recent paper [2, Theorem 2] Saddeek et al. have shown that, under certain appropriate conditions imposed on the mapping T, the entire sequence of KIM (1) converges weakly

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to a fixed point of T in a real Hilbert space setting. The results is then applied to a problem of fluid mechanics.

An interesting problem is how to appropriately modify the KIM (1) so as to have strong convergence? For this purpose, in this paper, inspired by He and Zhu [1], we introduce a modified Krasnoselskii iteration method MKIM (2) below with strong convergence (by boundary point method) for finding the minimum norm solution of nonlinear equation of the form $S_{h(x)}(x) = 0$, where $S_{h(x)}$ is a nonlinear mapping of C into itself and h is a function of C into $[0, 1]$ defined below. Furthermore, we apply this result to the stationary problem of seepage theory. The results obtained in this paper represent an extension as well as refinement of some earlier theorems of [2].

2. Preliminaries

Let C be a nonempty, closed and convex subset of X and let T be a mapping of C into itself.

In the sequel we use $F(T)$ to denote the nonempty set of fixed points of T, $\rightarrow$ ($\rightharpoonup$) to denote strong (weak) convergence, and $\mathfrak{M}(x_n)$ to denote the set of weak cluster points of (x_n) (i. e., $\mathfrak{M}(x_n) = \{x : \exists(x_{n_k}) \subset (x_n) : x_{n_k} \rightharpoonup x\}$). Denote by $proj_C(x)$ the metric projection mapping from X onto C (i. e. $proj_C(x) = \{y \in X : \|y - x\| = \inf_{x\in C}\|x - z\|\}$).

The projection mapping is characterized as follows (see, for example, [12]):

Proposition 2.1. *Given $x \in X$ and $z \in C$. Then $z = proj_C(x)$ if and only if*

$$\langle x - z, y - z\rangle \leqslant 0, \quad \forall y \in X.$$

The notions $U(x;\delta)$ and ∂C are used to denote, respectively, the spherical neighborhood of x of radius $\delta > 0 : U(x;\delta) = \{y \in X : \|y - x\| < \delta\}$ and the boundary of C (whenever C is closed): $\partial C = \{x \in X : U(x;\delta) \cap (X - C) \neq \phi \forall \delta > 0\}$.

Since $F(T)$ is a nonempty, closed and convex subset of X (see, for example, [13]), so there exists a unique $\hat{x} \in F(T)$ satisfies the following:

$$\|\hat{x}\| = \min\{\|x\| : x \in F(T)\}.$$

That is, $\hat{x}$ is the minimum norm fixed point of T. In other words, $\hat{x}$ is the metric projection of the origin onto $F(T)$, that is, $\hat{x} = proj_C(0)$.

Definition 2.1 (*see, for example, [14–16]*). For any $x, y \in C$ the mapping $T : C \rightarrow C$ is said to be as follows:

(i) pseudomonotone, if it is bounded and if for every sequence $\{x_n\} \subset C$ such that

$$x_n \rightharpoonup x \text{ and } \limsup_{n\to\infty}\langle Tx_n, x_n - x\rangle \leqslant 0,$$

imply that

$$\liminf_{n\to\infty}\langle Tx_n, x_n - y\rangle \geqslant \langle Tx, x - y\rangle;$$

(ii) coercive, if

$$\langle Tx, x\rangle \geqslant \rho(\|x\|)\|x\|, \lim_{\xi\to+\infty}\rho(\xi) = +\infty;$$

(iii) Lipschitzian, if there exists $L > 0$ such that

$$\|Tx - Ty\| \leqslant L\|x - y\|;$$

(iv) potential, if

$$\int_0^1 (\langle T(t(x+y), x+y\rangle - \langle T(tx), x\rangle))dt = \int_0^1 \langle T(x+ty), y\rangle dt;$$

(v) demiclosed at 0, if for every sequence $\{x_n\} \subset C$ the assumptions

$$x_n \rightharpoonup x \text{ and } Tx_n \rightarrow 0, \text{ as } n \rightarrow \infty$$

imply that

$$x \in C \text{ and } Tx = 0.$$

3. A MKIM by boundary point method

In order to state our algorithm we introduce, as in [1], $h : C \rightarrow [0, 1]$ by

$$h(x) = \inf\{\alpha \in [0, 1] : \alpha x \in C\}, \quad \forall x \in C.$$

The function $h(x)$ is well defined because C is closed and convex.

In [1] it has been noted that if $0 \in C$, then $h(x) = 0$ for all $x \in C$ and if $0 \notin C$, then $h(x)x \in \partial C$ and $h(x) > 0$ for every $x \in C$ (for a contradiction, suppose that $h(x)x \notin \partial C$ in the case where $0 \notin C$, it is easy to verify that $h(x)x$ is an inner point in C, there exists sufficiently small $\delta > 0$ such that $U(h(x)x;\delta) \subset C$, we have $\left[h(x) - \frac{\delta}{2}\right]x \in C$. This contradicts the definition of $h(x)$. Notice that C is closed and convex, hence, $h(x)x \in \partial C$). For more details of the properties of $h(x)$, the reader is refereed to [1].

Let $T : C \rightarrow C$ be a self mapping. Then the MKIM by boundary point is given by $x_0 = x \in C$, and

$$x_{n+1} = (1 - \tau h(x_n))x_n + \tau T_\tau x_n, \quad n \geqslant 0, \tag{2}$$

where $\tau \in (0, 1)$, $T_\tau = (1 - \tau)I + \tau T$ and $\sum_{n=0}^{\infty} h(x_n) = \infty$.

Remark 3.1.

(i) If $0 \notin C$ and $h(x_n) = 1$, then the MKIM $\{x_n\}$ given by (2) is exactly the KIM corresponding to the associated mapping T_τ.
(ii) If $0 \in C$ (i. e., $h(x_n) = 0$ *for all* $n \geqslant 0$), then the MKIM $\{x_n\}$ given by (2) is exactly the Picard iteration corresponding to the associated mapping $I + \tau T_\tau$.
(iii) Since C is closed and convex, it follows from the definition of $h(x)$ that, for any given $x \in C$, $\gamma x \in C$ holds for every $\gamma \in [h(x)x, 1]$, which imply that $\{x_n\} \subset C$ is guaranteed.
(iv) In the case where $0 \notin C$, calculating the value $h(x_n)$ implies determining $h(x_n)x_n$ (a boundary point of C), so our modification method is called boundary point method.

Lemma 3.1 (*see, for example [17,18]*). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leqslant (1 - \omega_n)a_n + \omega_n b_n + c_n, \quad \forall n \geqslant 0,$$

where, $\omega_n \subset (0, 1)$. If $\sum_{n=0}^{\infty}\omega_n = \infty$, either $\limsup_{n\to\infty} b_n \leqslant 0$ or $\sum_{n=0}^{\infty}|\omega_n b_n| < \infty$ and $\sum_{n=0}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n = 0$.