Journal of the Egyptian Mathematical Society (2014) **22**, 501–510

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

The meshless local Petrov-Galerkin method for simulating unsteady incompressible fluid flow ☆

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Received 22 February 2013; revised 14 March 2013; accepted 8 October 2013
Available online 5 December 2013

KEYWORDS

MLS;
MLPG;
Improved Gaussian's weight;
Unsteady incompressible fluid flow

Abstract This article presents a numerical algorithm using the Meshless Local Petrov-Galerkin (MLPG) method for the incompressible Navier–Stokes equations. To deal with time derivatives, the forward time differences are employed yielding the Poisson's equation. The MLPG method with the moving least-square (MLS) approximation for trial function is chosen to solve the Poisson's equation. In numerical examples, the local symmetric weak form (LSWF) and the local unsymmetric weak form (LUSWF) with a classical Gaussian weight and an improved Gaussian weight on both regular and irregular nodes are demonstrated. It is found that LSWF1 with a classical Gaussian weight order 2 gives the most accurate result.

1. Introduction

Incompressible Navier–Stokes flow in two dimensions is one of several major problems in fluid mechanics that have been extensively studied both theoretically and numerically. In general, the formulation of primitive variables is popularly employed for the incompressible Navier–Stokes equation but it has a limitation in approximating the velocity and the pressure. The finite volume method (FVM) and finite element method (FEM) have been widely applied to solve the incompressible Navier–Stokes flow problems. However, it is well-known that these methods depend strongly on the mesh properties. In computing problems with irregular complex geometries using these methods, mesh generation is a far more time-consuming and expensive task than solution of the partial differential equations (PDEs), particularly in three dimensional (3D) cases. To overcome such a problem, meshless methods, a new numerical method class have been developed.

Meshless methods were established with the objective of eliminating the requirement of mesh generation step, which is time-consuming and burdensome, in FEM. Owing to these

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Peer review under responsibility of Egyptian Mathematical Society.

ELSEVIER Production and hosting by Elsevier

☆ The research is financed by: This research is partially supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

http://dx.doi.org/10.1016/j.joems.2013.10.002

reasons, meshless methods have received much attention as a number of meshless methods have been introduced by different authors. These include smooth particle hydrodynamics (SPH) [1,2], diffuse element method (DEM) [3], element-free Galerkin (EFG) [4], reproducing kernel particle method (RKPM) [5], finite point method (FPM) [6], partition of unity method (PU) [7], boundary node method (BNM) [8], local boundary integral equation (LBIE) [9], meshless local Petrov-Galerkin method (MLPG) [10], meshless regular local boundary integral equation (MRLBIE) [11], finite cloud method (FCM) [12], point interpolation method (PIM) [13], least-squares collocation meshless method (LSCM) [14], etc. The meshless local Petrov-Galerkin (MLPG) method is a truly meshless method, which requires no elements or background cells, for either the interpolation or the integration purposes. The concept of MLPG was first proposed by Atluri and Zhu [10], and later discussed in depth in Atluri and Shen [15]. The most significant difference between this method and the finite element method or any other meshless method is that the local weak forms are generated on overlapping local sub-domains, instead of using the global weak form. Integration of the weak form is performed in local sub-domains with simple geometrical shapes, therefore no elements or background cells are necessary either for interpolation purposes or for integration purposes. The MLPG approach is also different from the truly meshless method based on the local boundary integral equation (LBIE) method, because there are no singular integrals in the MLPG method. This method is characterized as meshless since distributed nodal points, covering the domain of interest, are employed.

Remarkable successes of the MLPG method in computational mechanics have been reported in recent years. The first article applying MLPG method to compute convection-diffusion and incompressible flow problems was by Lin and Atluri [16]. In their work, two kinds of upwind schemes were constructed to overcome oscillations produced by convection term. They applied upwind schemes to solve the incompressible flow problem based on the primitive variable formulation and added the perturbation term to continuity equation to satisfy the Babŭka-Brezzi condition. But when these schemes were applied to compute the high Reynolds number problems, the parameter of perturbation term was difficult to determine and it also suffered from the convergent difficulty. Wu et al. [17] applied MLPG to solve incompressible flow problems with vorticity-stream function method without addressing the stability problem. One year later, they applied MLPG to solve two-dimensional (2D) incompressible fluid flow and heat transfer problems with benchmark solutions. The streamline upwind Petrov-Galerkin method is applied to overcome oscillation velocity field and mixed formulation is employed to satisfy the Babŭka-Brezzi condition. The results show that SUPG method gives a convergent solution for high Reynolds number. Sanyasiraju and Chandhini [18] developed a local RBF gridfree scheme to solve unsteady incompressible Navier–Stokes equations for primitive variables. This novel fractional step algorithm has been proposed to achieve velocity-pressure decoupling, in which it has been validated over various problems.

In the present paper, the meshless local Petrov-Galerkin method with MLS interpolation scheme is applied to develop an algorithm for solving the unsteady incompressible Navier–Stokes flow problem.

2. The moving least-square (MLS) approximation for trial function

The moving least-square (MLS) is one of these interpolation schemes with a reasonable accuracy. Consider a sub-domain $\Omega_{\mathbf{x}}$, which is defined as the neighborhood of a point $\mathbf{x}$ and denoted as domain of definition of MLS approximation for the trial function at point $\mathbf{x}$. To approximate the distribution of function $u^n(\mathbf{x}) = u(\mathbf{x}, t_n)$ in $\Omega_{\mathbf{x}}$, over a number of randomly located nodes $\mathbf{x}_i$, $i = 1, 2, \ldots, N$. The moving least-squares (MLS) approximation $u_h^n(\mathbf{x})$ of u^n, $\forall \mathbf{x} \in \Omega_{\mathbf{x}}$, can be defined by

$$u_h^n = \mathbf{p}^T(\mathbf{x})\mathbf{a}^n(\mathbf{x}), \qquad \forall \mathbf{x} \in \Omega_{\mathbf{x}}, \tag{1}$$

where $\mathbf{p}(\mathbf{x})$ is a vector of basis function

$$\mathbf{p}^T(\mathbf{x}) = [p_1(\mathbf{x}), p_2(\mathbf{x}), \ldots, p_m(\mathbf{x})],$$

where m is the number of the basis functions. Usually the complete monomial basis is used to ensure the consistency of the approximations, whereby different types of the polynomials may be used. Depending on the problem, other type of functions may also be employed in order to enhance the solutions. For a two-dimensional (2D) case used in this paper, the complete monomial basis are defined as follows:

- Linear basis

$$\mathbf{p}^T(\mathbf{x}) = [1,\ x,\ y],$$

- Quadratic basis

$$\mathbf{p}^T(\mathbf{x}) = [1,\ x,\ y,\ x^2,\ xy,\ y^2],$$

where $\mathbf{x} = (x, y) \in \mathbb{R}^2$ and the term of the complete 2D basis may be obtained by employing the Pascal triangle. For the polynomial basis, the total number of terms is related to the order of the basis by expression $m = \frac{(l+1)(l+2)}{2}$ with l as the order of the basis. The vector $\mathbf{a}^n(\mathbf{x})$ contains the unknown coefficients

$$\mathbf{a}^n(\mathbf{x}) = [a_1^n(\mathbf{x}),\ a_2^n(\mathbf{x}),\ a_3^n(\mathbf{x}),\ \ldots,\ a_m^n(\mathbf{x})]^T,$$

which are the functions of $\mathbf{x}$, i.e. they have to be calculated for each point $\mathbf{x}$. The vector $\mathbf{a}^n(\mathbf{x})$ is determined by means of the discrete weighted L_2 norm, defined as follows:

$$J(\mathbf{a}^n(\mathbf{x})) = \sum_{i=1}^{N} w_i(\mathbf{x})[\mathbf{p}^T(\mathbf{x}_i)\mathbf{a}^n(\mathbf{x}) - \hat{u}_i^n]^2, \tag{2}$$

where $w_i(\mathbf{x})$ is a weight function associated with the node i, $w_i(\mathbf{x}) > 0$ for all $\mathbf{x}$ in the support of $w_i(\mathbf{x})$, $\mathbf{x}_i$ denotes the values of $\mathbf{x}$ at node i, N is a number of nodes in $\Omega_{\mathbf{x}}$ for which $w_i(\mathbf{x}) > 0$. Here it should be noted that $\hat{u}_i^n$, $i = 1, 2, \ldots, N$ in Eq. (2) are the fictitious nodes and not the actual nodes of unknown trial function $u_h^n(\mathbf{x})$. The minimization of $J(\mathbf{a}^n(\mathbf{x}))$ leads to the following system of equations

$$A(\mathbf{x})\mathbf{a}^n(\mathbf{x}) = B(\mathbf{x})\hat{\mathbf{u}}^n, \tag{3}$$

where,

$$A = \sum_{i=1}^{N} w_i(\mathbf{x})\mathbf{p}(\mathbf{x}_i)\mathbf{p}^T(\mathbf{x}_i)$$

the matrix B is defined as

$$B = [w_1(\mathbf{x})\mathbf{p}(\mathbf{x}_1),\ w_2(\mathbf{x})\mathbf{p}(\mathbf{x}_2),\ w_3(\mathbf{x})\mathbf{p}(\mathbf{x}_3),\ \ldots,\ w_N(\mathbf{x})\mathbf{p}(\mathbf{x}_N)],$$