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ORIGINAL ARTICLE

# Finding all real roots of a polynomial by matrix algebra and the Adomian decomposition method

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**Abstract** In this paper, we put forth a combined method for calculation of all real zeroes of a polynomial equation through the Adomian decomposition method equipped with a number of developed theorems from matrix algebra. These auxiliary theorems are associated with eigenvalues of matrices and enable convergence of the Adomian decomposition method toward different real roots of the target polynomial equation. To further improve the computational speed of our technique, a nonlinear convergence accelerator known as the Shanks transform has optionally been employed. For the sake of illustration, a number of numerical examples are given.

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## 1. Introduction

Finding the roots of a polynomial equation has been among the oldest problems of mathematics. The solution of quadratics was known to the Arab and Persian scholars of the early Middle Ages, for example Omar Khayyam [1]. The cubic polynomial equation was first solved systematically by Cardano in mid-16th century. Soon afterward, the solution to quadratics was formulated [2]. In the early 19th century, Abel and Galios ingeniously proved that there exists no general formula for zeroes of a polynomial equation of degree five or higher. This is nowadays referred to as the Abel's impossibility theorem [3]. Since then, iterative schemes began to arise, of which mention can be made of the Newton–Raphson method of the 17th century, Bernoulli's method of the 18th, and Graeffe's method of the early 19th century. A superabundance of new algorithms has been emerged in the mathematical literature since the 20th century especially due to the advent of electronic computers [4]. For an extensive account on the history and progress of polynomial root-finding see [5–9] and the references therein.

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It is the objective of this paper to postulate a polynomial equation zero-finder by synergistic combination of the Adomian decomposition method and ideas from matrix algebra. Advantageous use of the companion matrix concept and the Gershgorin circle theorem will be made. In the final section, a number of numerical examples are included for the sake of illustration.

## 2. The Adomian decomposition method

For the ease of the reader, who is new to this method, we briefly review the basics of the Adomian decomposition method (ADM) in this section.

To illustrate the ADM, consider the following general functional equation:

$$u - N(u) = f, \tag{1}$$

where $N$ is a nonlinear operator, which maps a Hilbert space $H$ into itself, $f$ is a given function and $u$ designates an unknown function. The ADM decomposes the solution $u$ as an infinite summation $u = \sum_{i=0}^{\infty} u_i$ and $N$ as $N(u) = \sum_{i=0}^{\infty} A_i$, where $A_i$ are called the Adomian polynomials [10]:

$$A_i = A_i(u_0, u_1, \ldots, u_i) = \frac{1}{i!}\frac{d^i}{d\lambda^i} N\left(\sum_{k=0}^{\infty} u_k \lambda^k\right)\Bigg|_{\lambda=0}. \tag{2}$$

By letting $u_0 = f$, the ADM permits the following recursive relation to generate components of the solution,

$$\begin{cases} u_0 = f, \\ u_{i+1} = A_i; \quad i \geqslant 0. \end{cases} \tag{3}$$

The convergence and reliability of the ADM have been ascertained in prior works (e.g. [11,12]). In [13], Fatoorehchi and Abolghasemi have developed a completely different algorithm to generate the Adomian polynomials of any desired nonlinear operators mainly based on string functions and symbolic programming. For more background on the ADM and its applications, see [14–22] and the references mentioned therein.

## 3. The proposed method

Suppose that we are after the roots of the following polynomial equation,

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0. \tag{4}$$

Without loss of generality, we can convert Eq. (4) to its monic polynomial equation analog as,

$$Q(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_2 x^2 + b_1 x + b_0 = 0. \tag{5}$$

By definition, the companion matrix associated with $Q(x)$ reads,

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -b_0 \\ 1 & 0 & \cdots & 0 & -b_1 \\ 0 & 1 & \cdots & 0 & -b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -b_{n-1} \end{bmatrix}. \tag{6}$$

Denote by $eig()$ and $roots()$ the operators returning eigenvalues and zeroes of their matrix and polynomial arguments, respectively.

It holds true that,

$$eig(A) = roots(Q(x)). \tag{7}$$

In view of Eq. (7), the problem of zero finding for our polynomial equation is converted to a problem of determining the eigenvalues of a companion matrix.

Before we proceed, we need to state a few theorems that will come in handy in the sequel.

**Definition 3.1.** Let $A$ be a complex $n$-by-$n$ matrix, with entries $a_{ij}$. Let $R_i = \sum_{j\neq i}^{n} |a_{ij}|$, for $i \in \{1, \ldots, n\}$, be the sum of absolute values of the non-diagonal entries in the $i$th row. Also, let $D(a_{ii}, R_i)$ be the closed disk centered at $a_{ii}$ with radius $R_i$ Such a disk is dubbed as Gershgorin disk.

**Theorem 3.1** (*Gershgorin circle theorem*). *Every eigenvalue of $A$ lies within at least one of the Gershgorin disks $D(a_{ii}, R_i)$.*

**Proof.** For brevity, we exclude the proof and refer the reader to [23,24]. □

**Theorem 3.2.** *Let $A$ and $B$ be $n \times n$ matrices, $I$ represent identity matrix in $n$ dimensions, $\alpha$ denote a real number, and $eig()$ stand for an operator returning an eigenvalue of its matrix argument. If $B = A + \alpha I$, then it holds that $eig(B) = eig(A) + \alpha$.*

**Proof.** Let $eig(A) = \lambda$. This necessitates $\det(A - \lambda I) = 0$. Replacing $A$ with its equivalent gives $\det(B - \alpha I - \lambda I) = 0$ or obviously $\det(B - (\lambda + \alpha)I) = 0$. This asserts that the quantity $\lambda + \alpha$ is an eigenvalue for the matrix $B$ or in other words $eig(B) = eig(A) + \alpha$. □

**Theorem 3.3.** *Denote by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of an $n$-by-$n$ matrix $A$. It holds true that $trace(A) = \sum_{i=1}^{n} \lambda_i$.*

**Proof.** Due to space limitation, we suffice to refer the reader to [25] for the proof of this theorem. □

**Theorem 3.4.** *The characteristic polynomials of two similar matrices are identical.*

**Proof.** Suppose $A$ and $B$ be $n$-by-$n$ and similar to each other. Since $A$ and $B$ are similar, i.e. $A \sim B$, it is essential that $A = QBQ^{-1}$ for some invertible matrix $Q$. Take $a(x) = \det(A - xI)$, $b(x) = \det(B - xI)$ as the characteristic equations of $A$ and $B$, respectively. Hence, $a(x) = \det(QBQ^{-1} - xI)$. It follows that,

$$\begin{aligned} a(x) &= \det\left(-x\left[I - \frac{1}{x}QBQ^{-1}\right]\right) \\ &= (-x)^n \det\left(I - \frac{1}{x}QBQ^{-1}\right). \end{aligned} \tag{8}$$

Take $C = -1/xQB$ and $D = Q^{-1}$. Applying the Sylvester determinant theorem [26], we have

$$a(x) = (-x)^n \det(I + CD) = (-x)^n \det(I + DC). \tag{9}$$

So,

$$a(x) = (-x)^n \det\left(I + Q^{-1}\left[-\frac{1}{x}QB\right]\right). \tag{10}$$

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