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ORIGINAL ARTICLE

On integral operators for certain classes of *p*-valent functions associated with generalized multiplier transformations

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Abstract In this paper, we study new generalized integral operators for the classes of p-valent functions associated with generalized multiplier transformations.

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Introduction

Let $\mathcal{A}(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j \quad (p \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. We write $\mathcal{A}(1) = \mathcal{A}$.

For two functions f and g, analytic in U, we say that the function f is subordinate to g, if there exists a Schwarz function w, i.e, $w \in \mathcal{A}(p)$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. This subordination is usually denoted by $f(z) \prec g(z)$. It is well-known that if the function g is univalent in U, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [1]). If f is subordinate to g, then g is superordinate to f.

Let $\mathcal{P}_k(p, \rho)$ be the class of functions $g(z)$ analytic in U satisfying the properties $g(0) = p$ and

$$\int_0^{2\pi} \left| \frac{\Re\{g(z)\} - \rho}{p - \rho} \right| d\theta \leqslant k\pi, \tag{1.2}$$

where $z = re^{i\theta}$, $k \geqslant 2$ and $0 \leqslant \rho < p$. This class was introduced by Aouf [2, with $\lambda = 0$].

We note that:

(i) $\mathcal{P}_k(1, \rho) = \mathcal{P}_k(\rho) (k \geqslant 2,\ 0 \leqslant \rho < 1)$(see [3]);
(ii) $\mathcal{P}_k(1, 0) = \mathcal{P}_k\ (k \geqslant 2)$ (see [4,5]);
(iii) $\mathcal{P}_2(p, \rho) = \mathcal{P}(p, \rho)\ (0 \leqslant \rho < p, p \in \mathbb{N})$, where $\mathcal{P}(p, \rho)$ is the class of functions with positive real part greater than α (see [6]);

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(iv) $\mathcal{P}_2(p,0)=\mathcal{P}(p)(p\in\mathbb{N})$, where $\mathcal{P}(p)$ is the class of functions with positive real part (see [6]).

The classes $\mathcal{R}_k(p,\rho)$ and $\mathcal{V}_k(p,\rho)$ are related to the class $\mathcal{P}_k(p,\rho)$ and can be defined as

$$f\in\mathcal{R}_k(p,\rho)\iff\frac{zf'(z)}{f(z)}\in\mathcal{P}_k(p,\rho)\quad(z\in U),\tag{1.3}$$

and

$$f\in\mathcal{V}_k(p,\rho)\iff\frac{(zf'(z))'}{f'(z)}\in\mathcal{P}_k(p,\rho)\quad(z\in U).\tag{1.4}$$

Using the concept of subordination, Aouf [2], with $\alpha=0$ introduced the class $\mathcal{P}[p,A,B]$ as follows:

For A and B, $-1\leqslant B<A\leqslant1$, a function h analytic in U with $h(0)=p$ belongs to the class $\mathcal{P}[p,A,B]$ if h is subordinate to $p\frac{1+Az}{1+Bz}$.

Let $\mathcal{P}_k[p,A,B](k\geqslant2,-1\leqslant B<A\leqslant1)$ denote the class of p-valent analytic functions h that are represented by

$$h(z)=\left(\frac{k}{4}+\frac{1}{2}\right)h_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)h_2(z)\ (z\in U;\ h_1,h_2\in\mathcal{P}[p,A,B]).\tag{1.5}$$

Now we define the following classes $\mathcal{R}_k[p,A,B]$ and $\mathcal{V}_k[p,A,B]$ of the class $\mathcal{A}(p)$ for $k\geqslant2$ and $-1\leqslant B<A\leqslant1$ as follows:

$$\mathcal{R}_k[p,A,B]=\left\{f:f\in\mathcal{A}(p)\text{ and }\frac{zf'(z)}{f(z)}\in\mathcal{P}_k[p,A,B],\ z\in U\right\},\tag{1.6}$$

and

$$\mathcal{V}_k[p,A,B]=\left\{f:f\in\mathcal{A}(p)\text{ and }\frac{(zf'(z))'}{f'(z)}\in\mathcal{P}_k[p,A,B],\ z\in U\right\}\tag{1.7}$$

Obviously, we know that

$$f(z)\in\mathcal{V}_k[p,A,B]\iff\frac{zf'(z)}{p}\in\mathcal{R}_k[p,A,B].\tag{1.8}$$

We note that $\mathcal{P}_k[1,A,B]=\mathcal{P}_k[A,B]$, $\mathcal{R}_k[1,A,B]=\mathcal{R}_k[A,B]$ and $\mathcal{V}_k[1,A,B]=\mathcal{V}_k[A,B]$ (see [7]).

Prajapat [8] defined a generalized multiplier transformation operator $\mathcal{J}_p^m(\lambda,\ell):\mathcal{A}(p)\to\mathcal{A}(p)$, as follows:

$$\mathcal{J}_p^m(\delta,\ell)f(z)=z^p+\sum_{j=p+1}^{\infty}\left(\frac{p+\ell+(j-p)\delta}{p+\ell}\right)^m a_kz^k$$
$$(\delta\geqslant0;\ \ell>-p;\ p\in\mathbb{N};\ m\in\mathbb{Z}=\{0,\pm1,\ldots\};\ z\in U).\tag{1.9}$$

It is readily verified from (1.3) that

$$\delta z(\mathcal{J}_p^m(\delta,\ell)f(z))'=(\ell+p)\mathcal{J}_p^{m+1}(\delta,\ell)f(z)-[\ell+p(1-\delta)]\mathcal{J}_p^m(\delta,\ell)f(z)\ (\delta>0).\tag{1.10}$$

By specializing the parameters m, δ, ℓ and p, we obtain the following operators studied by various authors:

(i) $\mathcal{J}_p^m(\delta,\ell)f(z)=I_p^m(\delta,\ell)f(z)$ ($\ell\geqslant0,p\in\mathbb{N},\delta\geqslant0$ and $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$) (see [9]);
(ii) $\mathcal{J}_p^m(1,\ell)f(z)=I_p(m,\ell)f(z)$ ($\ell\geqslant0,p\in\mathbb{N}$ and $m\in\mathbb{N}_0$) (see [10,11]);
(iii) $\mathcal{J}_p^m(\delta,0)f(z)=D_{\delta,p}^mf(z)$($\delta\geqslant0,p\in\mathbb{N}$ and $m\in\mathbb{N}_0$) (see [12]);
(iv) $\mathcal{J}_p^m(1,0)f(z)=D_p^mf(z)$ ($m\in\mathbb{N}_0$ and $p\in\mathbb{N}$) (see [13–15]);
(v) $\mathcal{J}_p^{-m}(\delta,\ell)f(z)=J_p^m(\delta,\ell)f(z)$ ($\ell\geqslant0,\delta\geqslant0$, $p\in\mathbb{N}$ and $m\in\mathbb{N}_0$) (see [16–18]);
(vi) $\mathcal{J}_p^{-m}(1,1)f(z)=D^mf(z)$ ($m\in\mathbb{Z}$) (see [19]);
(vii) $\mathcal{J}_1^m(1,\ell)f(z)=I_\ell^mf(z)$ ($\ell\geqslant0$ and $m\in\mathbb{N}_0$) (see [20,21]);
(viii) $\mathcal{J}_1^m(\delta,0)f(z)=D_\delta^mf(z)$ ($\delta\geqslant0$ and $m\in\mathbb{N}_0$) (see [22]);
(ix) $\mathcal{J}_1^m(1,0)f(z)=D^mf(z)$ ($m\in\mathbb{N}_0$) (see [23]);
(x) $\mathcal{J}_1^{-m}(\delta,0)f(z)=I_\delta^{-m}f(z)$ ($\delta\geqslant0$ and $m\in\mathbb{N}_0$) (see [24,25]);
(xi) $\mathcal{J}_1^{-m}(1,1)f(z)=I^mf(z)$ ($m\in\mathbb{N}_0$) (see [26]).

Let us consider the integral operators:

$$\mathcal{F}_{p,\delta,\ell}^{n,m}(z)=\int_0^z pt^{p-1}\left(\frac{\mathcal{J}_p^m(\delta,\ell)f_1(t)}{t^p}\right)^{\alpha_1}\cdots\left(\frac{\mathcal{J}_p^m(\delta,\ell)f_n(t)}{t^p}\right)^{\alpha_n}dt\tag{1.11}$$

and

$$\mathcal{G}_{p,\delta,\ell}^{n,m}(z)=\int_0^z pt^{p-1}\left(\frac{(\mathcal{J}_p^m(\delta,\ell)f_1(t))'}{pt^{p-1}}\right)^{\beta_1}\cdots\left(\frac{(\mathcal{J}_p^m(\delta,\ell)f_n(t))'}{pt^{p-1}}\right)^{\beta_n}dt,\tag{1.12}$$

where $f_i(z)\in\mathcal{A}(p)$ and α_i, $\beta_i>0$ for $i=\{1,2,\ldots,n\}$.

We note that:

(i) $\mathcal{F}_{p,\delta,\ell}^{n,0}(z)=F_p(z)$ and $\mathcal{G}_{p,\delta,\ell}^{n,0}(z)=G_p(z)$ (see [27,28]);
(ii) $\mathcal{F}_{p,\delta,0}^{n,m}(z)=F_n(z)$ (see [29,30]).

Also, we note that

(i)
$$\mathcal{F}_{p,\delta,\ell}^{n,m}(z)=\mathcal{I}_{p,\delta,\ell}^{n,m}(z)=\int_0^z pt^{p-1}\left(\frac{I_p^m(\delta,\ell)f_1(t)}{t^p}\right)^{\alpha_1}\cdots\left(\frac{I_p^m(\delta,\ell)f_n(t)}{t^p}\right)^{\alpha_n}dt$$
$$(\ell\geqslant0;\ \delta\geqslant0;\ p\in\mathbb{N};\ m\in\mathbb{N}_0)\tag{1.13}$$

and

$$\mathcal{G}_{p,\delta,\ell}^{n,m}(z)=G_{p,\delta,\ell}^{n,m}(z)=\int_0^z pt^{p-1}\left(\frac{\left(I_p^m(\delta,\ell)f_1(t)\right)'}{pt^{p-1}}\right)^{\beta_1}\cdots\left(\frac{\left(I_p^m(\delta,\ell)f_n(t)\right)'}{pt^{p-1}}\right)^{\beta_n}dt$$
$$(\ell\geqslant0;\ \delta\geqslant0;\ p\in\mathbb{N};\ m\in\mathbb{N}_0);\tag{1.14}$$

(ii)
$$\mathcal{F}_{p,\delta,0}^{n,m}(z)=\mathcal{D}_{p,\delta}^{n,m}(z)=\int_0^z pt^{p-1}\left(\frac{D_{\delta,p}^mf_1(t)}{t^p}\right)^{\alpha_1}\cdots\left(\frac{D_{\delta,p}^mf_n(t)}{t^p}\right)^{\alpha_n}dt$$
$$(\delta\geqslant0;\ p\in\mathbb{N};\ m\in\mathbb{N}_0)\tag{1.15}$$

and