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ORIGINAL ARTICLE

Fixed point theorems in fuzzy metric spaces ☆

M.A. Ahmed *

Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt

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Abstract In this paper, we state and prove some common fixed point theorems in fuzzy metric spaces. These theorems generalize and improve known results (see [1]).

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1. Introduction

In 1965, the theory of fuzzy sets was investigated by Zadeh [2]. In 1981, Heilpern [3] first introduced the concept of fuzzy contractive mappings and proved a fixed point theorem for these mappings in metric linear spaces. His result is a generalization of the fixed point theorem for point-to-set maps of Nadler [4]. Therefore, several fixed point theorems for types of fuzzy contractive mappings have appeared (see, for instance [1,5–9]).

In this paper, we state and prove some common fixed point theorems in fuzzy metric spaces. These theorems generalize and improve known results (see [1]).

* Tel.: +20 882317965.
E-mail address: mahmed68@yahoo.com

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2. Basic preliminaries

The definitions and terminologies for further discussions are taken from Heilpern [3]. Let (X,d) be a metric linear space. A **fuzzy set** in X is a function with domain X and values in $[0,1]$. If A is a fuzzy set and $x \in X$, then the function-value $A(x)$ is called the **grade of membership** of x in A. The collection of all fuzzy sets in X is denoted by $\Im(X)$.

Let $A \in \Im(X)$ and $\alpha \in [0,1]$. The **α-level set** of A, denoted by A_α, is defined by

$$A_\alpha = \{x : A(x) \geqslant \alpha\} \quad \text{if} \quad \alpha \in (0,1], \quad A_0 = \overline{\{x : A(x) > 0\}},$$

whenever $\overline{B}$ is the closure of set (nonfuzzy) B.

Definition 2.1. A fuzzy set A in X is an **approximate quantity** iff its α-level set is a nonempty compact convex subset (nonfuzzy) of X for each $\alpha \in [0,1]$ and $sup_{x\in X}A(x) = 1$.

The set of all approximate quantities, denoted by $W(X)$, is a subcollection of $\Im(X)$.

Definition 2.2. Let $A, B \in W(X)$, $\alpha \in [0,1]$ and $CP(X)$ be the set of all nonempty compact subsets of X. Then

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$$p_\alpha(A,B) = \inf_{x\in A_\alpha, y\in B_\alpha} d(x,y), \quad \delta_\alpha(A,B) = \sup_{x\in A_\alpha, y\in B_\alpha} d(x,y) \quad \text{and}$$
$$D_\alpha(A,B) = H(A_\alpha, B_\alpha),$$

where H is the **Hausdorff metric** between two sets in the collection $CP(X)$. We define the following functions

$$p(A,B) = \sup_\alpha p_\alpha(A,B), \quad \delta(A,B) = \sup_\alpha \delta_\alpha(A,B) \quad \text{and}$$
$$D(A,B) = \sup_\alpha D_\alpha(A,B).$$

It is noted that p_α is nondecreasing function of α.

Definition 2.3. Let A, $B \in W(X)$. Then A is said to be **more accurate** than B (or B includes A), denoted by $A \subset B$, iff $A(x) \leqslant B(x)$ for each $x \in X$.

The relation $\subset$ induces a partial order on $W(X)$.

Definition 2.4. Let X be an arbitrary set and Y be a metric linear space. F is said to be a **fuzzy mapping** iff F is a mapping from the set X into $W(Y)$, i.e., $F(x) \in W(Y)$ for each $x \in X$.

The following proposition is used in the sequel.

Proposition 2.1. ([4]).*If A, $B \in CP(X)$ and $a \in A$, then there exists $b \in B$ such that $d(a,b) \leqslant H(A,B)$.*

Following Beg and Ahmed [10], let (X,d) be a metric space. We consider a subcollection of $\Im(X)$ denoted by $W^*(X)$. Each fuzzy set $A \in W^*(x)$, its α-level set is a nonempty compact subset (nonfuzzy) of X for each $\alpha \in [0,1]$. It is obvious that each element $A \in W(X)$ leads to $A \in W^*(X)$ but the converse is not true.

The authors [10] introduced the improvements of the lemmas in Heilpern [3] as follows.

Lemma 2.1. *If $\{x_0\} \subset A$ for each $A \in W^*(X)$ and $x_0 \in X$, then $p_\alpha(x_0,B) \leqslant D_\alpha(A,B)$ for each $B \in W^*(X)$.*

Lemma 2.2. *$p_\alpha(x,A) \leqslant d(x,y) + p_\alpha(y,A)$ for all x, $y \in X$ and $A \in W^*(X)$.*

Lemma 2.3. *Let $x \in X$, $A \in W^*(X)$ and $\{x\}$ be a fuzzy set with membership function equal to a characteristic function of the set $\{x\}$. Then $\{x\} \subset A$ if and only if $p_\alpha(x,A) = 0$ for each $\alpha \in [0,1]$.*

Lemma 2.4. *Let (X,d) be a complete metric space, F: $X \to W^*(X)$ be a fuzzy map and $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.*

Remark 2.1. It is clear that Lemma 2.4 is a generalization of corresponding lemma in Arora and Sharma [1] and Proposition 3.2 in Lee and Cho [7].

Let Ψ be the family of real lower semi-continuous functions F: $[0,\infty)^6 \to R$, $R :=$ the set of all real numbers, satisfying the following conditions:

(ψ_1) F is non-increasing in 3rd, 4th, 5th, 6th coordinate variable,
(ψ_2) there exists $h \in (0,1)$ such that for every u, $v \geqslant 0$ with
(ψ_{21}) $F(u,v,v,u,u+v,0) \leqslant 0$ or (ψ_{22}) $F(u,v,u,v,0,u+v) \leqslant 0$, we have $u \leqslant h\, v$, and
(ψ_3) $F(u,u,0,0,u,u) > 0$ for all $u > 0$.

3. Main results

In 2000, Arora and Sharma [1] proved the following result.

Theorem 3.1. *Let (X,d) be a complete metric space and T_1, T_2 be fuzzy mappings from X into $W(X)$. If there is a constant q, $0 \leqslant q < 1$, such that, for each x, $y \in X$,*

$$D(T_1(x), T_2(y)) \leqslant q \max\{d(x,y), p(x,T_1(x)), p(y,T_2(y)), p(x,T_2(y)), p(y,T_1(x))\},$$

then there exists $z \in X$ such that $\{z\} \subset T_1(z)$ and $\{z\} \subset T_2(z)$.

Remark 3.1. If there is a constant q, $0 \leqslant q < 1$, such that, for each x, $y \in X$,

$$D(T_1(x), T_2(y)) \leqslant q \max\{d(x,y), p(x,T_1(x)), p(y,T_2(y))\}, \quad (1)$$

then the conclusion of Theorem 3.1 remains valid. This result is considered as a special case of Theorem 3.1.

Beg and Ahmed [10] generalized Theorem 3.1 as follows.

Theorem 3.2. *Let (X,d) be a complete metric space and T_1, T_2 be fuzzy mappings from X into $W^*(X)$. If there is a $F \in \Psi$ such that, for all x, $y \in X$,*

$$F(D(T_1(x), T_2(y)), d(x,y), p(x,T_1(x)), p(y,T_2(y)), p(x,T_2(y)), p(y,T_1(x))) \leqslant 0, \quad (2)$$

then there exists $z \in X$ such that $\{z\} \subset T_1(z)$ and $\{z\} \subset T_2(z)$.

Widely inspired by a paper of Tas et al. [11], we give another different generalization of Theorem 3.1 with contractive condition (1) as follows.

Theorem 3.3. *Let (X,d) be a complete metric space and T_1, T_2 be fuzzy mappings from X into $W^*(X)$. Assume that there exist c_1, c_2, $c_3 \in [0,\infty)$ with $c_1 + 2c_2 < 1$ and $c_2 + c_3 < 1$ such that, for all x, $y \in X$,*

$$D^2(T_1(x), T_2(y)) \leqslant c_1 \max\{d^2(x,y), p^2(x,T_1(x)), p^2(y,T_2(y))\} + c_2 \max\{p(x,T_1(x))p(x,T_2(y)), p(y,T_1(x)) p(y,T_2(y))\} + c_3 p(x,T_2(y))p(y,T_1(x)). \quad (3)$$

Then there exists $z \in X$ such that $\{z\} \subset T_1(z)$ and $\{z\} \subset T_2(z)$.

Proof. Let x_0 be an arbitrary point in X. Then by Lemma 2.4, there exists an element $x_1 \in X$ such that $\{x_1\} \subset T_1(x_0)$. For $x_1 \in X$, $(T_2(x_1))_1$ is nonempty compact subset of X. Since $(T_1(x_0))_1$, $(T_2(x_1))_1 \in CP(X)$ and $x_1 \in (T_1(x_0))_1$, then Proposition 2.1 asserts that there exists $x_2 \in (T_2(x_1))_1$ such that $d(x_1,x_2) \leqslant D_1(T_1(x_0), T_2(x_1))$. So, we obtain from the inequality $D(A,B) \geqslant D_\alpha(A,B)$ $\forall \alpha \in [0,1]$ that

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