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ORIGINAL ARTICLE

Some fixed point results without monotone property in partially ordered metric-like spaces

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Abstract The purpose of this paper is to obtain the fixed point results for F-type contractions which satisfies a weaker condition than the monotonicity of self-mapping of a partially ordered metric-like space. A fixed point result for F-expansive mapping is also proved. Therefore, several well known results are generalized. Some examples are included which illustrate the results.

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1. Introduction and preliminaries

Ran and Reurings [1] and Nieto and Lopez [2,3] obtained the existence of fixed points of a self-mapping of a metric space equipped with a partial order. The fixed point results in spaces equipped with a partial order can be applied in proving existence and uniqueness of solutions for matrix equations as well as for boundary value problems of ordinary differential equations, integral equations, fuzzy equations, of problems in L-spaces, etc. (see [1–11]). The results of Ran and Reurings [1] and Nieto and Lopez [2,3] were generalized by several authors (see, e.g., [4,5,8,12–17]).

In all these papers, the condition of monotonicity with respect to the partial order defined on space is required. Following is a typical result among these.

Theorem 1 ([1,2]). *Let $(X,\sqsubseteq)$ be a partially ordered set which is directed (upward or downward) and let d be a metric on X such that (X,d) is a complete metric space. Let $f:X\to X$ be a mapping such that the following conditions hold:*

(i) *f is monotone (nondecreasing or nonincreasing) on X with respect to "$\sqsubseteq$";*
(ii) *there exists $x_0\in X$ such that $x_0\sqsubseteq fx_0$ or $fx_0\sqsubseteq x_0$;*
(iii) *there exists $k\in(0,1)$ such that $d(fx,fy)\leqslant kd(x,y)$ for all $x,y\in X$ with $y\sqsubseteq x$;*
(iv) (a) *f is continuous, or*

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(b) *if a nondecreasing sequence* $\{x_n\}$ *converges to* $x \in X$, *then* $x_n \sqsubseteq x$ *for all* n.

Then, f has a fixed point $x^* \in X$.

Recently, the fixed point results on partially ordered sets are investigated via a weaker property than the monotonicity of f (see [8,13,18,19]). We state following facts from these papers.

Let $(X, \sqsubseteq)$ be a partially ordered set and $x, y \in X$. If x, y are comparable (i.e., $x \sqsubseteq y$ or $y \sqsubseteq x$ holds), then we will write $x \asymp y$.

Lemma 2 [18]. *Consider the following properties for a self-map f on a partially ordered set* $(X, \sqsubseteq)$:

1. *f is monotone (nondecreasing or nonincreasing), i.e.,* $x \sqsubseteq y \Rightarrow fx \sqsubseteq fy$ *for all* $x, y \in X$ *or* $y \sqsubseteq x \Rightarrow fx \sqsubseteq fy$ *for all* $x, y \in X$;
2. $x \asymp y \Rightarrow fx \asymp fy$ for $x, y \in X$;
3. $x \asymp fx \Rightarrow fx \asymp ffx$ for $x \in X$.

Then $1 \Rightarrow 2 \Rightarrow 3$. *The reverse implications do not hold in general*.

On the other hand, Matthews [20] introduced the notion of partial metric space as a part of the study of denotational semantics of data flow network. In this space, the usual metric is replaced by partial metric with an interesting property that the self-distance of any point of space may not be zero. Further, Matthews showed that the Banach contraction principle is valid in partial metric space and can be applied in program verification.

Very recently, Amini-Harandi [21] generalized the partial metric spaces by introducing the metric-like spaces and proved some fixed point theorems in such spaces. In [22], Wardowski introduced a new concept of an F-contraction and proved a fixed point theorem which generalizes Banach contraction principle in a different way than in the known results from the literature in complete metric spaces. In this paper, we consider a more generalized type of F-contractions and prove some common fixed point theorems for such type of mappings in metric-like spaces. We generalize the result of Wardowski [22], Matthews [20], Ran and Reurings [1], Nieto and Lopez [2], and the recent result of Đorić et al. [18] by proving the fixed point results for $F-g-$weak contractions in metric-like spaces equipped with a partial order. Results of this paper are new not only in the setting of metric-like spaces but also in the setting of metric and partial metric spaces.

First, we recall some definitions and facts about partial metric and metric-like spaces.

Definition 1 [20]. A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ ($\mathbb{R}^+$ stands for nonnegative reals) such that, for all $x, y, z \in X$:

(p1) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$;
(p2) $p(x,x) \leqslant p(x,y)$;
(p3) $p(x,y) = p(y,x)$;
(p4) $p(x,y) \leqslant p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X. A sequence $\{x_n\}$ in (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x_n,x)$. A sequence $\{x_n\}$ in (X,p) is called p-Cauchy sequence if there exists $\lim_{n,m\to\infty} p(x_n,x_m)$ and is finite. (X,p) is said to be complete if every p-Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Definition 2 [21]. A metric-like on a nonempty set X is a function $\sigma : X \times X \to \mathbb{R}^+$ such that, for all $x, y, z \in X$:

(σ1) $\sigma(x,y) = 0$ implies $x = y$;
(σ2) $\sigma(x,y) = \sigma(y,x)$;
(σ3) $\sigma(x,y) \leqslant \sigma(x,z) + \sigma(z,y)$.

A metric-like space is a pair (X,σ) such that X is a nonempty set and σ is a metric-like on X. Note that, a metric-like satisfies all the conditions of metric except that $\sigma(x,x)$ may be positive for $x \in X$. Each metric-like σ on X generates a topology τ_σ on X whose base is the family of open σ-balls

$$B_\sigma(x,\epsilon) = \{y \in X : |\sigma(x,y) - \sigma(x,x)| < \epsilon\}, \quad \text{for all } x \in X \text{ and } \epsilon > 0.$$

A sequence $\{x_n\}$ in X converges to a point $x \in X$ if and only if $\lim_{n\to\infty} \sigma(x_n,x) = \sigma(x,x)$. Sequence $\{x_n\}$ is said to be σ-Cauchy if $\lim_{n,m\to\infty} \sigma(x_n,x_m)$ exists and is finite. The metric-like space (X,σ) is called σ-complete if for each σ-Cauchy sequence $\{x_n\}$, there exists $x \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n,x) = \sigma(x,x) = \lim_{m,n\to\infty} \sigma(x_n,x_m).$$

Note that every partial metric space is a metric-like space, but the converse may not be true.

Example 1 [21]. Let $X = \{0,1\}$ and $\sigma : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma(x,y) = \begin{cases} 2, & \text{if } x = y = 0; \\ 1, & \text{otherwise.} \end{cases}$$

Then (X,σ) is a metric-like space, but it is not a partial metric space, as $\sigma(0,0) \nleq \sigma(0,1)$.

Example 2. Let $X = \mathbb{R}, k \geqslant 0$ and $\sigma : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma(x,y) = \begin{cases} 2k, & \text{if } x = y = 0; \\ k, & \text{otherwise.} \end{cases}$$

Then (X,σ) is a metric-like space, but for $k > 0$, it is not a partial metric space, as $\sigma(0,0) \nleq \sigma(0,1)$.

Example 3. Let $X = \mathbb{R}^+$ and $\sigma : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma(x,y) = \begin{cases} 2x, & \text{if } x = y; \\ \max\{x,y\}, & \text{otherwise.} \end{cases}$$

Then (X,σ) is a metric-like space, it is not a partial metric space, as $\sigma(1,1) = 2 \nleq \sigma(0,1) = 1$.

Definition 3. If a nonempty set X is equipped with a partial order "$\sqsubseteq$" such that (X,σ) is a metric-like space, then the $(X,\sigma,\sqsubseteq)$ is called a partially ordered metric-like space. A subset