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ORIGINAL ARTICLE

A comparative study of numerical methods for solving the generalized Ito system

M.A. Mohamed [a], **M.Sh. Torky** [b,*]

[a] *Faculty of Science, Suez Canal University Ismailia, Egypt*
[b] *The High Institute of Administration and Computer, Port Said University, Egypt*

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Abstract This paper is devoted to the numerical comparison of methods applied to solve the generalized Ito system. Four numerical methods are compared, namely, the Laplace decomposition method (LDM), the variation iteration method (VIM), the homotopy perturbation method (HPM) and the Laplace decomposition method with the Pade approximant (LD–PA) with the exact solution.

2000 MATHEMATICS SUBJECT CLASSIFICATION:
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1. Introduction

In recent years 1999, the variational iteration method (VIM) was proposed by He in [1–6]. This method is now widely used by many researchers to study linear and nonlinear partial differential equations. The method introduces a reliable and efficient process for a wide variety of scientific and engineering applications, linear or nonlinear, homogeneous or inhomogeneous, equations and systems of equations as well. Many authors [7–11] that this method is more powerful than existing techniques such as the Adomian method [12,16], perturbation method, etc. showed it. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists; otherwise a few approximations can be used for numerical purposes. Another important advantage is that the VIM method is capable of greatly reducing the size of calculation while still maintaining high accuracy of the numerical solution. Moreover, the power of the method gives it a wider applicability in handling a huge number of analytical and numerical applications. Many authors for different cases have obtained some exact and numerical solutions of the generalized Ito system (see [17–20]).

Consider the generalized Ito system [21]:

$$u_t = v_x, \quad v_t = -2v_{xxx} - 6(uv)_x + aww_x + bpw_x + cwp_x + dpp_x + fw_x + gp_x,$$
$$w_t = w_{xxx} + 3uw_x, \quad p_t = p_{xxx} + 3up_x. \tag{1}$$

The general exact solution of above system is:

* Corresponding author. Tel.: +20 1224012802.
E-mail addresses: dr_magdy_ahmed53@yahoo.cm (M.A. Mohamed), mohamedtorky75@yahoo.com (M.Sh. Torky).
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$$u(x,t) = \frac{2-\beta}{3} - 2\tanh(x - t\beta)^2, v(x,t) = \frac{\beta^2-2}{3} + 4\beta\tanh(x - t\beta)^2,$$

$$w(x,t) = \frac{-bf + ag - b^2c_0 + adc_0}{a(b-c)} + \frac{2\left(-\frac{b^2c\sqrt{-8a\beta-5a\beta^2}}{a\sqrt{-bc+ad}} + \frac{bd\sqrt{-8a\beta-5a\beta^2}}{\sqrt{-bc+ad}}\right)\tanh(x-t\beta)}{-bc+ad},$$

$$P(x,t) = c_0 - \frac{2\sqrt{-8a\beta-5a\beta^2}\tanh(x-t\beta)}{\sqrt{-bc+ad}}, \quad (2)$$

For simplicity, we take: $a = -37,\ b = 2,\ c = \frac{1}{2},\ d = -1,\ f = 2,\ g = 2,\ c_0 = -1,\ \beta \to \frac{1}{4}$. We have:

$$\mathrm{u(x,t)} = \frac{7}{12} - 2\tanh\left(x - \frac{t}{4}\right)^2, \quad \mathrm{v(x,t)} = -\frac{7}{48} + \frac{1}{2}\tanh\left(x - \frac{t}{4}\right)^2,$$
$$\mathrm{w(x,t)} = -\frac{36}{37} + \frac{1}{6}\tanh\left(x - \frac{t}{4}\right), \quad \mathrm{p(x,t)} = 4 + \frac{37}{12}\tanh\left(x - \frac{t}{4}\right), \quad (3)$$

with initial conditions:

$$\mathrm{u(x,0)} = \frac{7}{12} - 2\tanh(x)^2, \mathrm{v(x,0)} = -\frac{7}{48} + \frac{1}{2}\tanh(x)^2,$$
$$\mathrm{w(x,0)} = -\frac{36}{37} + \frac{1}{6}\tanh(x), \mathrm{p(x,0)} = 4 + \frac{37}{12}\tanh(x). \quad (4)$$

The aim of this paper is to use the Laplace decomposition method (LDM), the variation iteration method (VIM), Homotopy perturbation method (HPM) and the Pade approximant (LD–PA) to find the numerical solution of Eq. (1), compare our obtained results with the exact solution, and compute the error.

2. Methods and its applications

2.1. The variational iteration method [3–8]

For the purpose of illustration of the methodology to the proposed method, using variational iteration method [22–26], we write a system in an operator form as:

$$L_t u + R_1(u,v,w,p) + N_1(u,v,w,p) = g_1,$$
$$L_t v + R_2(u,v,w,p) + N_2(u,v,w,p) = g_2,$$
$$L_t w + R_3(u,v,w,p) + N_3(u,v,w,p) = g_3,$$
$$L_t p + R_4(u,v,w,p) + N_4(u,v,w,p) = g_4, \quad (5)$$

with initial data

$$u(x,0) = f_1(x), \quad v(x,0) = f_2(x),$$
$$w(x,0) = f_3(x), \quad p(x,0) = f_4(x), \quad (6)$$

where L_t is considered a first-order partial differential operator, R_j, $1 \leqslant j \leqslant 4$, and N_j, $1 \leqslant j \leqslant 4$, are linear and nonlinear operators respectively, and g_1, g_2 and g_3 are source terms. In what follows we give the main steps of He's variational iteration method in handling scientific and engineering problems. The system (5) can be written as:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1\{Lu_n(\tau) + R_1(\breve{u},\breve{v},\breve{w},\breve{p}) + N_1(\breve{u},\breve{v},\breve{w},\breve{p}) - g_1(\tau)\}d\tau,$$
$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda_2\{Lv_n(\tau) + R_2(\breve{u},\breve{v},\breve{w},\breve{p}) + N_2(\breve{u},\breve{v},\breve{w},\breve{p}) - g_2(\tau)\}d\tau,$$
$$w_{n+1}(x,t) = w_n(x,t) + \int_0^t \lambda_3\{Lw_n(\tau) + R_3(\breve{u},\breve{v},\breve{w},\breve{p}) + N_3(\breve{u},\breve{v},\breve{w},\breve{p}) - g_3(\tau)\}d\tau,$$
$$p_{n+1}(x,t) = p_n(x,t) + \int_0^t \lambda_4\{Lp_n(\tau) + R_4(\breve{u},\breve{v},\breve{w},\breve{p}) + N_4(\breve{u},\breve{v},\breve{w},\breve{p}) - g_4(\tau)\}d\tau, \quad (7)$$

where λ_j, $1 \leqslant j \leqslant 4$, are general Lagrange multipliers [7], which can be identified optimally via the variational theory, and u_n, v_n, w_n and p_n are restricted variations which means $\delta u_n = 0$, $\delta v_n = 0$, $\delta w_n = 0$ and $\delta p_n = 0$. It is required first to determine the Lagrange multipliers λ_j that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(x,t)$, $v_{n+1}(x,t)$, $w_{n+1}(x,t)$, $p_{n+1}(x,t)$, $n \geqslant 0$, of the solutions $u(x,t)$, $v(x,t)$, $w(x,t)$ and $p(x,t)$ will follow immediately upon using the Lagrange multipliers obtained and by using selected functions u_0, v_0, and w_0. The initial values are usually used for the selected zeroth approximations. With the Lagrange multipliers λj determined, then several approximations $u_j(x,t)$, $v_j(x,t)$, $w_j(x,t)$, $p_j(x,t)$ $j \geqslant 0$, can be determined. Consequently, the solutions are given by

$$u(x,t) = \lim_{n\to\infty\partial} u_n(x,t), \quad v(x,t) = \lim_{n\to\infty\partial} v_n(x,t),$$
$$w(x,t) = \lim_{n\to\infty\partial} p_n(x,t), \quad p(x,t) = \lim_{n\to\infty\partial} p_n(x,t). \quad (8)$$

2.2. The Laplace decomposition method (LDM) [27]

In this section, Laplace decomposition method [27–29] is applied to the system of partial differential Eq. (1). The method consists of first applying the Laplace transformation to both sides of (1)

$$\pounds[L_t u] = \pounds[g_1] + \pounds[R_1(u,v,w,p) + N_1(u,v,w,p)],$$
$$\pounds[L_t v] = \pounds[g_2] + \pounds[R_2(u,v,w,p) + N_2(u,v,w,p)],$$
$$\pounds[L_t w] = \pounds[g_3] + \pounds[R_3(u,v,w,p) + N_3(u,v,w,p)],$$
$$\pounds[L_t p] = \pounds[g_4] + \pounds[R_4(u,v,w,p) + N_4(u,v,w,p)]. \quad (9)$$

Using the formulas of the Laplace transform, we get

$$s\pounds[u] - u(0) = \pounds[g_1] + \pounds[R_1(u,v,w,p) + N_1(u,v,w,p)],$$
$$s\pounds[v] - v(0) + \pounds[R_2(u,v,w,p) + N_2(u,v,w,p)],$$
$$s\pounds[w] - w(0) = \pounds[g_3] + \pounds[R_3(u,v,w,p) + N_3(u,v,w,p)],$$
$$s\pounds[p] - p(0) = \pounds[g_4] + \pounds[R_4(u,v,w,p) + N_4(u,v,w,p)]. \quad (10)$$

In the Laplace decomposition method, we assume the solution as an infinite series, given as follows:

$$u = \sum_n^\infty u_n, \quad v = \sum_n^\infty v_n, \quad w = \sum_n^\infty w_n, \quad p = \sum_n^\infty p_n, \quad (11)$$

where the terms u_n are to be recursively computed. In addition, the linear and nonlinear terms R_1, R_2, R_3, R_4 and N_1, N_2, N_3, N_4 are decompose as an infinite series of Adomian polynomials (see [12–16]):