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ORIGINAL ARTICLE

Numerical study of partial differential equations to estimate thermoregulation in human dermal regions for temperature dependent thermal conductivity

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Abstract The paper deals with the temperature distribution in multi-layered human skin and subcutaneous tissues (SST). The model suggests the solution of parabolic heat equation together with the boundary conditions for the temperature distribution in SST by assuming the thermal conductivity as a function of temperature.

The model formulation is based on singular non-linear boundary value problem and has been solved using finite difference method. The numerical results were found similar to clinical and computational results.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 92BXX; 92CXX; 92C35; 92C50; 46N60

1. Introduction

The unstable ambient temperature plays an important role for the disturbance in human thermoregulatory system. The effect of surrounding temperature makes its way via dermal layers and leads to hyperthermia and hypothermia to the body core and tissue necrosis to the body peripherals. Several researchers studied the distribution of temperature in the human body organs in relation to several environment temperatures. The mathematical model for temperature distribution in the human dermal layers can be represented as a boundary value problem. In this study, the domain is consisting of dermal layers and the formulation is based on the differential equation of heat conduction as

$$k\frac{d^2T}{d^2r} + \frac{2k}{r}\frac{dT}{dr} + Q = 0 \tag{1}$$

and the boundary conditions are

$$\lim_{r\to 0^+}\frac{dT}{dr} = 0, \quad \text{and} \quad -k\left(\frac{dT}{dr}\right)_{r=R} = E(T_H - T_a) \tag{2}$$

where r is the radial distance from the core of the domain, R – the radius of the domain, E – the evaporation term, T_a – the ambient temperature, T_H – the periphery temperature,

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Q – the heat production per unit volume and k is the thermal conductivity at the dermal regions.

Various mathematical models were formulated to study the effect of environmental temperatures on human dermal regions. Our group had also developed numerous models in this direction. Few models are demonstrated by Khanday and Saxena [1–3], but the thermal conductivity was assumed as constant or function of space parameter to describe the thermoregulation in biological tissues. The main purpose of this study is to estimate the temperature profiles at the dermal regions with respect to changes in ambient temperature and thermal conductivity as a function of temperature.

Thron [4] studied the above model to estimate the temperature distribution in human head and suggested that if there is no singularity in the above differential equation, then the solution is given by

$$T(r) = T_a + \frac{QR^2}{6k}\left[1 + \frac{2k}{ER} - \left(\frac{r}{R}\right)^2\right] \tag{3}$$

In addition, he calculated the temperature distribution by assuming additional heat sources while the cooling of blood at periphery by the Eq. (3) with

$$Q = Q_0 + Q_b \tag{4}$$

where $Q_b = Vs(T_1 - T)$, Q_0 is the heat production of tissue, V is volume of the flow of blood in unit time, T_1 is the deep temperature of the core and $s = 0.9\ \text{cal}/^\circ\text{C cm}^3$.

Richardson and Whitelaw [5] predicted the temperature profiles in the biological tissues by keeping skin surface as functions of temperature. Flesch [6] estimated the temperature distribution using the heat Eq. (1) by assuming a heat generation rate as an explicit function of the radial distance and an implicit function of the environment temperature. Khanday and Saxena [2] calculated the mass and temperature distribution at multi-layered skin and sub-dermal tissues by using variational finite element method with respect to various environmental temperatures. Also they studied the thermostat phenomenon of brain tissue and estimated the cold stress at multi-layered human head with respect to ambient temperatures.

The present work is an attempt to study the distribution of temperature at deep dermal layers for heterogeneous thermal conductivity as a function of temperature.

2. Mathematical formulation of the model

The mathematical modelling for the estimation of temperature distribution in human body has gained interest among many researchers and our group has published many papers by taking into account various parameters and other physiological aspect of the domain under study. The heat transfer in biological tissues was considered initially by Pennes [7] and further elaborated by other researchers as well. The role of temperature to the changes in thermal conductivity of the material has been incorporated by means of the term $k(T) = k_0(T - T_H)^n$. The mathematical model of the heat transfer in the human dermal regions has been considered by the following usual differential equation of heat conduction

$$\rho c\frac{\partial T}{\partial t} = k(T)\nabla^2 T + k'(T)\nabla \cdot T + Q \tag{5}$$

where ∇ is an operator determining the first order partial derivative of T with respect to three dimensional systems; ρ, c and k represent the density, specific heat of the tissue and thermal conductivity respectively.

In case of steady state processes, the above equation can be written as

$$k_0\frac{d^2T}{dr^2} + \frac{2k_0}{r}\frac{dT}{dr} + \frac{nk_0}{(T - T_H)}\left(\frac{dT}{dr}\right)^2 + \frac{Q}{(T - T_H)^n} = 0 \tag{6}$$

Using the fact that the heat generation term Q is a function of temperature T. Therefore, let $Q = q_1(37 - T)$ for some positive constant q_1.

Thus, singular boundary value problem determining the heat conduction at dermal layers has been established as follows

$$k_0\frac{d^2T}{dr^2} + \frac{2k_0}{r}\frac{dT}{dr} + \frac{nk_0}{(T - T_H)}\left(\frac{dT}{dr}\right)^2 + \frac{q_1(37 - T)}{(T - T_H)^n} = 0 \tag{7}$$

$$\left.\frac{dT}{dr}\right|_{r=0} = 0, \quad T(R) = T_H \tag{8}$$

where Eq. (8) is the boundary conditions.

In order to non-dimensionalize the above boundary value problem, we make use of the following transformations

$$y = T - T_H \quad \text{and } t = r/R,\ 0 < t < 1, \tag{9}$$

The following system of equations results from the transformation

$$k_0\frac{d^2y}{dt^2} + \frac{2k_0}{t}\frac{dy}{dt} + \frac{nk_0}{y}\left(\frac{dy}{dt}\right)^2 + \frac{q_1(37 - y - T_H)R^2}{y^n} = 0,$$

$$0 < t < 1, \left.\frac{dy}{dt}\right|_{t=0} = 0, \quad y(1) = 0 \tag{10}$$

Define the following substitutions,

$$c(t) = t^2 \quad \text{and } f(t, y, cy') = \frac{nk_0}{y}\left(\frac{dy}{dt}\right)^2 + \frac{q_1(37 - y - T_H)R^2}{y^n}$$

we have,

$$\begin{aligned} &\frac{1}{c(t)}[c(t)y'(t)]' + f(t, y, cy') = 0 \\ &y'(0) = 0, \quad y(1) = 0 \end{aligned} \tag{11}$$

The solution of the singular non-linear boundary value problem (11) exists and is unique as discussed by Celik [8]. The solution was approximated by means of the finite difference method.

3. Solution and interpretation of the model

The distribution of temperature in human dermal regions can be sought to solve the boundary value problem (10) numerically.

The finite difference method will be applied to solve (10) as follows:

Dividing (0, 1) into p subintervals with the length of each subinterval as $h = 1/p$, then by the central differences, the above equation for $i = 0$, can be written as

$$2k_0y_1 + \frac{q_1(37 - y_0 - T_H)R^2h^2}{y_0^n} - 2k_0y_0 = 0 \tag{12}$$

and for $i = 1, 2, 3, \ldots, p - 1$