Journal of the Egyptian Mathematical Society (2013) **21**, 241–247

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Attractivity of two nonlinear third order difference equations

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Received 30 December 2012; revised 26 February 2013; accepted 12 March 2013
Available online 23 April 2013

KEYWORDS

Difference equation;
Periodic solution;
Attractivity;
Asymptotically stable

Abstract The aim of this work is to investigate the global attractivity, periodic nature, oscillation and the boundedness of all admissible solutions of the difference equations

$$x_{n+1} = \frac{A - Bx_{n-1}}{\pm C + Dx_{n-2}}, \quad n = 0, 1, \ldots$$

where A, B are nonnegative real numbers, C, D are positive real numbers and $\pm C + Dx_{n-2} \neq 0$ for all $n \geqslant 0$.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 39A11

1. Introduction

Difference equations appear naturally as discrete analogues and numerical solutions of differential equations and delay differential equations having applications in biology, ecology, physics, etc, [1].

The study of nonlinear rational difference equations of higher order is of paramount importance, since we still know so little about such equations.

El-Owaidy et al. [2] investigated the global attractivity of the difference equation

$$x_{n+1} = \frac{\alpha - \beta x_{n-1}}{\gamma + x_n}, \quad n = 0, 1, \ldots$$

where, α, β, γ are non-negative real numbers and $\gamma + x_n \neq 0$ for all $n \geqslant 0$.

Xiu-Mei et al. [3] investigated the global attractivity of the negative solutions of the nonlinear difference equation

$$x_{n+1} = \frac{1 - x_{n-k}}{A + Dx_n}, \quad n = 0, 1, \ldots$$

where $A \in (-\infty, 0)$, k is a positive integer and $A + Dx_n \neq 0$ for all $n \geqslant 0$.

Wan-Sheng et al. [4] studied the attractivity of the nonlinear delay difference equation

$$x_{n+1} = \frac{a - bx_{n-k}}{A + x_n}, \quad n = 0, 1, \ldots$$

where $a \geqslant 0$, b, $A > 0$, $k \in \{1, 2, \ldots\}$ and $A + x_n \neq 0$ for all $n \geqslant 0$.

In this paper, we study the global attractivity of the difference equations

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Peer review under responsibility of Egyptian Mathematical Society.

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http://dx.doi.org/10.1016/j.joems.2013.03.009

$$x_{n+1} = \frac{A - Bx_{n-1}}{\pm C + Dx_{n-2}}, \quad n = 0, 1, \ldots \tag{1.1}$$

where A, B are nonnegative real numbers, C, D are positive real numbers and $\pm C + Dx_{n-2} \neq 0$ for all $n \geqslant 0$.

2. Preliminaries

Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots \tag{2.1}$$

Let f be a continuous function which maps some set J^{k+1} into J, where J is some interval of real numbers. It is easy to see that Eq. (2.1) has a unique solution $\{x_n\}_{n=-k}^{\infty}$ for the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in J$.

Definition 2.1. [5]

(1) An equilibrium point $\bar{x}$ for Eq. (2.1) is called locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, \ldots, x_0 \in J$ with $\sum_{i=-k}^{0}|x_i - \bar{x}| < \delta$ we have $|x_n - \bar{x}| < \epsilon$ for all $n \in \mathbb{N}$. Otherwise $\bar{x}$ is said to be unstable.
(2) The equilibrium point $\bar{x}$ of Eq. (2.1) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that for any initial conditions with $\sum_{i=-k}^{0}|x_i - \bar{x}| < \gamma$, we have $\lim_{n\to\infty} x_n = \bar{x}$.
(3) The equilibrium point $\bar{x}$ for Eq. (2.1) is called a global attractor if $x_{-k}, x_{-k+1}, \ldots, x_0 \in J$ always implies that $\lim_{n\to\infty} x_n = \bar{x}$.
(4) The equilibrium point $\bar{x}$ for Eq. (2.1) is called globally asymptotically stable if it is locally asymptotically stable and global attractor.

The linearized equation associated with Eq. (2.1) is

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \ldots, \bar{x}) y_{n-i}, \quad n = 0, 1, \ldots \tag{2.2}$$

The characteristic equation associated with Eq. (2.2) is

$$\lambda^{k+1} - \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \ldots, \bar{x}) \lambda^{k-i} = 0. \tag{2.3}$$

Theorem 2.2 [5]. *Assume that f is a C^1 function and let $\bar{x}$ be an equilibrium point of Eq. (2.1). Then the following statements are true:*

(1) *If all roots of Eq. (2.3) lie in the open disk $|\lambda| < 1$, then $\bar{x}$ is locally asymptotically stable.*
(2) *If at least one root of Eq. (2.3) has absolute value greater than one, then $\bar{x}$ is unstable.*

Theorem 2.3 [6]. *Consider the third-degree polynomial equation*

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0, \tag{2.4}$$

where a_0, a_1 and a_2 are real numbers. Then a necessary and sufficient condition that all roots of Eq. (2.4) to lie inside the open disk $|\lambda| < 1$ is

$$|a_2 + a_0| < 1 + a_1, \quad |a_2 - 3a_0| < 3 - a_1, \quad a_0^2 + a_1 - a_0a_2 < 1.$$

Theorem 2.4. *Consider the difference equation*

$$x_{n+1} = f(x_{n-1}, x_{n-2}), \quad n = 0, 1, \ldots \tag{2.5}$$

Let $[a, b]$ be an interval of real numbers and assume that

$f : [a, b] \times [a, b] \to [a, b]$

is a continuous function satisfying the following properties:

(1) $f(x, y)$ is non-decreasing in $x \in [a, b]$ for each $y \in [a, b]$ and $f(x, y)$ is non-increasing in $y \in [a, b]$ for each $x \in [a, b]$;
(2) If $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$f(m, M) = m \quad \text{and} \quad f(M, m) = M,$$

then $m = M$.*Then Eq. (2.5) has a unique equilibrium $\bar{x} \in [a, b]$ and every solution of (2.5) converges to $\bar{x}$.*

The change of variables $x_n = \frac{C}{D} y_n$ reduces Eq. (1.1) to the difference equation

$$y_{n+1} = \frac{p - qy_{n-1}}{\pm 1 + y_{n-2}}, \quad n = 0, 1, \ldots \tag{2.6}$$

where $p = \frac{AD}{C^2}$ and $q = \frac{B}{C}$.

In what follows, we will only consider the solutions corresponding to admissible initial conditions which will be called admissible solutions.

The stability is refereed to the set of admissible solutions.

3. The difference equation $y_{n+1} = \frac{p - qy_{n-1}}{1 + y_{n-2}}$

In this section we study the global attractivity of the difference equation

$$y_{n+1} = \frac{p - qy_{n-1}}{1 + y_{n-2}}, \quad n = 0, 1, \ldots \tag{3.1}$$

We can see that the equilibrium points of Eq. (3.1) are the zeros of the function

$$f_1(\bar{y}) = \bar{y}^2 + (1 + q)\bar{y} - p.$$

That is

$$\bar{y_1} = \frac{1}{2}(-(1 + q) + \sqrt{(1 + q)^2 + 4p})$$

and

$$\bar{y_2} = \frac{1}{2}(-(1 + q) - \sqrt{(1 + q)^2 + 4p}).$$

The linearized equation associated with Eq. (3.1) about $\bar{y_i}$, $i = 1, 2$ is

$$z_{n+1} + \frac{q}{1 + \bar{y_i}} z_{n-1} + \frac{\bar{y_i}}{1 + \bar{y_i}} z_{n-2} = 0, \quad n = 0, 1, \ldots \tag{3.2}$$

Its associated characteristic equation is

$$\lambda^3 + \frac{q}{1 + \bar{y_i}}\lambda + \frac{\bar{y_i}}{1 + \bar{y_i}} = 0. \tag{3.3}$$

Suppose that

$$g_i(\lambda) = \lambda^3 + \frac{q}{1 + \bar{y_i}}\lambda + \frac{\bar{y_i}}{1 + \bar{y_i}} = 0, \quad i = 1, 2.$$