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ORIGINAL ARTICLE

Solving BVPs with shooting method and VIMHP

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Abstract In this paper, a new method is applied for solving the nonlinear Boundary value problems. This method is a combination of shooting method and Variational Iteration Method Using He's Polynomials. As examples show, our proposed technique can overcome the difficulties that arise in both methods, and efficiency of this technique is approved.

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1. Introduction

Solving nonlinear differential equations are very important because they have the ability to model most phenomena in the world. So, scientist and researchers are interested finding the best way to determine the solution of nonlinear ODEs and PDEs. One group of differential equations is Boundary value problems [1] that can be solved by numerical methods. As we know, besides the ability of numerical techniques to find the solution of differential equations, they need huge computational work and are very time consuming. One numerical technique is shooting method that is used to solve BVPs. In this method by choosing the arbitrary value for derivatives of desired function in staring time and converting BVPs into IVPs, and then using numerical methods such as Runge-Kutta, the boundary value problem can be solved. Unfortunately, this procedure is not easy to apply. Therefore, it seems using analytical method can overcome this problem. But we encounter a problem in most of the analytical methods like Variational Iteration Method (VIM) [2–4], Homotopy perturbation Method (HPM) [5–7] and Adomian Decomposition Method (ADM) [8–10] for solving BVPs. In these methods, in order to get better results, we should choose the appropriate initial guess to start the recursive procedure, and this is difficult for BVPs. In this paper, we combine the VIMHP as a convergent and powerful method with shooting method to solve BVPs. This modification overcomes the mentioned difficulties that exist in numerical and analytical techniques for solving BVPs. The accuracy of the proposed method is approved via solving some examples and comparing the obtained results with the solutions of other methods.

2. Methodology

In following, the concepts of VIMHP, shooting method and our proposed method are presented:

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2.1. Variational iteration method

To illustrate the basic idea of VIM, at first consider the following nonlinear differential equation

$$L[u(r)] + N[u(r)] = g(r), \quad r > 0, \tag{1}$$

where $L = \frac{d^m}{dr^m}$, $m \in N$, is a linear operator as, N is a nonlinear operator and $g(r)$ is the source inhomogeneous term, subject to the initial conditions

$$u^{(k)}(0) = c_k, \quad k = 0, 1, 2, \ldots, m-1, \tag{2}$$

where c_k is a real number. According to the He's variational iteration method [11], we can construct a correction functional for (1) as follows:

$$u_{n+1}(r) = u_n(r) + \int_0^r \lambda(\tau)\{Lu_n(\tau) + N\widetilde{u_n}(\tau) - g(\tau)\}d\tau, \quad n \geqslant 0,$$

where λ is a general Lagrangian multiplier and can be identified optimally via variational theory. Here, we apply restricted variations to nonlinear term Nu, in this case the best value of multiplier we can be easily determined. Making the above functional stationary, noticing that $\delta\widetilde{u_n} = 0$,

$$\delta u_{n+1}(r) = \delta u_n(r) + \delta \int_0^r \lambda(\tau)\{Lu_n(\tau) - g(\tau)\}d\tau,$$

yields the following Lagrange multipliers,

$$\lambda = -1 \quad for\ m = 1,$$

$$\lambda = \tau - r, \quad for\ m = 2,$$

and in general,

$$\lambda = \frac{(-1)^m}{(m-1)!}(\tau - r)^{(m-1)}, \quad for\ m \geqslant 1.$$

The successive approximations $u_n(r)$, $n \geqslant 0$ of the solution $u(r)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. Consequently, the exact solution may be obtained as

$$u(r) = \lim_{n\to\infty} u_n(r).$$

2.2. Homotopy perturbation method

We know the essential idea of HPM [12–14] is to introduce a homotopy parameter, say p, which takes the values from 0 to 1. When $p = 0$, the system of equations is in sufficiently simplified form, which normally admits a rather simple solution. As p gradually increases to 1, the system goes through a sequence of "deformation", the solution of each stage is "close" to that at the previous stage of "deformation". Eventually at $p = 1$, the system takes the original form of equation and the final stage of "deformation" gives the desired solution. To illustrate the basic concept of HPM, consider the following nonlinear system of differential equations

$$A(\mathbf{U}) = f(\mathbf{r}), \qquad \mathbf{r} \in \Omega, \tag{3}$$

with boundary conditions

$$B\left(\mathbf{U}, \frac{\partial \mathbf{U}}{\partial n}\right) = 0, \qquad \mathbf{r} \in \Gamma,$$

where A is a differential operator, B is a boundary operator, $f(\mathbf{r})$ is a known analytic function, and Γ is the boundary of the domain Ω. Generally speaking the operator A can be divided into two parts L and N, where L is a linear, and N is a nonlinear operator. Therefore (3) can be rewritten as follows:

$$L(\mathbf{U}) + N(\mathbf{U}) - f(\mathbf{r}) = 0.$$

We construct a homotopy $\mathbf{V}(\mathbf{r}, p)$: $\Omega \times [0,1] \to R^n$, which satisfies

$$H(\mathbf{V}, p) = (1-p)[L(\mathbf{V}) - L(\mathbf{U}_0)] + p[A(\mathbf{V}) - f(\mathbf{r})] = 0, \quad p \in [0,1], \quad \mathbf{r} \in \Omega,$$

or equivalently,

$$H(\mathbf{V}, p) = L(\mathbf{V}) - L(\mathbf{U}_0) + pL(U_0) + p[N(V) - f(r)] = 0. \tag{4}$$

where U_0 is an initial approximation of (3). In this method, using the homotopy parameter p, we have the following power series presentation for $\mathbf{V}$,

$$\mathbf{V} = \mathbf{V}_0 + p\mathbf{V}_1 + p^2\mathbf{V}_2 + \ldots.$$

The approximate solution can be obtained by setting $p = 1$, i.e.

$$\mathbf{U} = \mathbf{U}_0 + \mathbf{U}_1 + \mathbf{U}_2 + \ldots. \tag{5}$$

The convergence of series (5) is discussed in [15]. The method considers the nonlinear term $N[\mathbf{V}]$ as

$$N(\mathbf{V}) = \sum_{i=0}^{+\infty} p^i H_i = H_0 + pH_1 + p^2H_2 + \cdots$$

where H_n's are the so-called He's polynomials [16], which can be calculated by using the formula

$$H_n(\mathbf{V}_0, \mathbf{V}_1, \cdots, \mathbf{V}_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(N\left(\sum_{i=0}^{n} p^i \mathbf{V}_i\right)\right)_{p=0}, \quad n = 0, 1, 2, \cdots$$

2.3. Variational iteration method using He's polynomials

To illustrate the basic idea of VIMHP, consider the following general differential equation:

$$L[u(r)] + N[u(r)] = g(r), \tag{6}$$

where L is a linear operator, N a nonlinear operator and $g(r)$ is the source inhomogeneous term. According to VIM, for $n \geqslant 0$ we can construct a correct functional as follows:

$$u_{n+1}(r) = u_n(r) + \int_0^r \lambda(\tau)\left\{\frac{d^m}{d\tau^m}u_n(\tau) + N[\widetilde{u_n}(\tau)] - g(\tau)\right\}d\tau, \tag{7}$$

where $\lambda(\tau) = \frac{(-1)^m}{(m-1)!}(\tau - r)^{(m-1)}$. Now, applying a series of the power of p and then using He's polynomials we have:

$$\sum_{n=0}^{\infty} p^n v_n = u_0 + p\int_0^r \lambda(\tau)\left[\frac{d^m}{dr^m}\left(\sum_{n=0}^{\infty} p^n v_n(\tau)\right) + N\left(\sum_{n=0}^{\infty} p^n v_n(\tau)\right)\right]d\tau - \int_0^r \lambda(\tau)g(\tau)d\tau = u_0 + p\int_0^r \lambda(\tau)\left[\frac{d^m}{dr^m}\left(\sum_{n=0}^{\infty} p^n v_n(\tau)\right) + \sum_{n=0}^{\infty} p^n H_n\right]d\tau - \int_0^r \lambda(\tau)g(\tau)d\tau, \tag{8}$$