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ORIGINAL ARTICLE

Contiguous relations for ${}_2F_1$ hypergeometric series

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Abstract Two Gauss functions are said to be contiguous if they are alike except for one pair of parameters, and these differ by unity. Contiguous relations are of great use in extending numerical tables of the function. In this paper we will introduce a new method for computing such types of relations.

1. Introduction

The study of hypergeometric series was launched many years ago by Euler, Gauss and Riemann; such series are the subject of considerable research. Hypergeometric series have a somewhat formidable notation, which takes a little time to get used to.

In 1812, Gauss presented to the Royal Society of Sciences at Göttingen his famous paper (Gauss, 1813) in which he considered the infinite series

$$1+\frac{a_1a_2}{1\cdot a_3}z+\frac{a_1(a_1+1)a_2(a_2+1)}{1\cdot 2\cdot a_3(a_3+1)}z^2+\frac{a_1(a_1+1)(a_1+2)a_2(a_2+1)(a_2+2)}{1\cdot 2\cdot 3\cdot a_3(a_3+1)(a_3+2)}z^3+\cdots \quad (1)$$

as a function of a_1, a_2, a_3, z, where it is assumed that $a_3 \neq 0, -1, -2, \ldots$, so that no zero factors appear in the denominators of the terms of the series. He showed that the series converges absolutely for $|z| < 1$, and for $|z| = 1$ when $Re(a_3 - a_1 - a_2) > 0$, gave its (contiguous) recurrence relations, and derived his famous formula

$$F(a_1, a_2; a_3; 1) = \frac{\Gamma(a_3)\Gamma(a_3-a_1-a_2)}{\Gamma(a_3-a_1)\Gamma(a_3-a_2)}, \quad Re(a_3-a_1-a_2) > 0 \quad (2)$$

for the sum of his series when $z = 1$ and $Re(a_3 - a_1 - a_2) > 0$.

Although Gauss used the notation $F(a_1, a_2, a_3, z)$ for his series, it is now customary to use $F[a_1, a_2; a_3; z]$ or either of the notations

$${}_2F_1(a_1, a_2; a_3; z), \quad {}_2F_1\left[\begin{matrix} a_1, a_2 \\ a_3 \end{matrix}; z\right] \quad (3)$$

for the series (and for its sum when it converges), because these notations separate the numerator parameters a_1, a_2 from the denominator parameter a_3 and the variable z. In view of Gauss' paper, his series is frequently called Gauss' series. However, since the special case $a_1 = 1$, $a_2 = a_3$ yields the geometric series

$$1 + z + z^2 + z^3 + \cdots \quad (4)$$

Gauss' series is also called (ordinary) hypergeometric series or the Gauss hypergeometric series. For more details about hypergeometric series and their contiguous relations, see [1–4].

Two hypergeometric functions with the same argument z are *contiguous* if their parameters a_1, a_2 and a_3 differ by integers. Gauss derived analogous relations between ${}_2F_1[a_1, a_2; a_3; z]$

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and any two contiguous hypergeometrics in which a parameter has been changed by ± 1. Rainville [5] generalized this to cases with more parameters.

Applications of contiguous relations range from the evaluation of hypergeometric series to the derivation of summation and transformation formulas for such series, they can be used to evaluate a hypergeometric function that is contiguous to a hypergeometric series which can be satisfactorily evaluated. Contiguous relations are also used to make a correspondence between Lie algebras and special functions. The correspondence yields formulas of special functions [6].

The 15 Gauss contiguous relations for ${}_2F_1[a_1, a_2; a_3; z]$ hypergeometric series imply that any three ${}_2F_1[a_1, a_2; a_3; z]$ series whose corresponding parameters differ by integers are linearly related (over the field of rational functions in the parameters). In [7], several properties of coefficients of these general contiguous relations were proved and then used to propose effective ways to compute contiguous relations. In [8], contiguous relations were used to establish and prove sharp inequalities between the Gaussian hypergeometric function and the power mean. These results extend known inequalities involving the complete elliptic integral and the hypergeometric mean. More details about contiguous relations and their application can be found in [9–14].

In this paper, we will extend the results obtained in [15], to prove different identities that relate between the contiguous functions of ${}_2F_1[a_1, a_2; a_3; z]$ hypergeometric functions. We will generalize the method of Theorem 1.1. of Vidúnas in [7], in which he summarizes some properties of the coefficients of contiguous relations. This method will be useful in computations and application of contiguous relations.

The paper is organized as follows: In Section 2, we introduce our method of computations; in Section 3 we introduce our main theorem in which we generalize the operators we defined in Section 2, while in Section 4, we use *Mathematica* to show how helpful is our main theorem in deriving contiguous function relations as well as to obtain any of their consequences.

2. Computations

Gauss defined as *contiguous* to ${}_2F_1[a_1, a_2; a_3; z]$ each of the six functions obtained by increasing or decreasing one of the parameters by unity [16, pp. 555–566]. Thus ${}_2F_1[a_1, a_2; a_3; z]$ is contiguous to the six functions

$${}_2F_1[a_1 \pm 1, a_2; a_3; z], \quad {}_2F_1[a_1, a_2 \pm 1; a_3; z] \quad \text{and} \quad {}_2F_1[a_1, a_2; a_3 \pm 1; z]$$

Gauss proved that between ${}_2F_1[a_1, a_2; a_3; z]$ and any two of its contiguous functions, there exists a linear relation with coefficients at most linear in z. These relationships are of great use in extending numerical tables of the function, since for one fixed value of z, it is necessary only to calculate the values of the function over two units in a_1, a_2 and a_3, and apply some recurrence relations in order to find the function values over a large range of values of a_1, a_2 and a_3 in this particular z-plane. A contiguous relation between any three contiguous hypergeometric functions can be found by combining linearly a sequence of Gauss contiguous relations.

In this section, we will introduce our method of computations from which we will be able to prove any type of contiguous relation, and for simplicity in the notation, let us introduce the following definition:

Definition 1. Let $\mathcal{A}_i^{\alpha_i} : X \to X$, $(i = 1, 2, 3)$, where X is the set of all Gauss' functions ${}_2F_1[a_1, a_2; a_3; z]$ with variable z, and parameters a_1, a_2 and a_3 such that $a_3 \neq 0, -1, -2, \ldots$, then

$$\mathcal{A}_1^{\alpha_1}(C[a_1, a_2, a_3]{}_2F_1[a_1, a_2; a_3; z]) = C[a_1 + \alpha_1, a_2, a_3]\ {}_2F_1[a_1 + \alpha_1, a_2; a_3; z] \tag{5}$$

$$\mathcal{A}_2^{\alpha_2}(C[a_1, a_2, a_3]\ {}_2F_1[a_1, a_2; a_3; z]) = C[a_1, a_2 + \alpha_2, a_3]\ {}_2F_1[a_1, a_2 + \alpha_2; a_3; z] \tag{6}$$

$$\mathcal{A}_3^{\alpha_3}(C[a_1, a_2, a_3]\ {}_2F_1[a_1, a_2; a_3; z]) = C[a_1, a_2, a_3 + \alpha_3]{}_2F_1[a_1, a_2; a_3 + \alpha_3; z] \tag{7}$$

where α_i, $i = 1, 2, 3$ are any integers, and $C[a_1, a_2, a_3]$ is an arbitrary constant function of a_1, a_2 and a_3 such that for any such operators

$$\mathcal{A}_i^{\alpha_i}\mathcal{A}_i^{-\alpha_i}(C[a_1, a_2, a_3]\ {}_2F_1[a_1, a_2; a_3; z]) = \mathcal{I}(C[a_1, a_2, a_3]\ {}_2F_1[a_1, a_2; a_3; z])$$

and $\mathcal{I}$ is the identity operator defined on X with

$$\mathcal{I}^k(C[a_1,a_2,a_3]\,{}_2F_1[a_1,a_2;a_3;z]) = \mathcal{I}(C[a_1,a_2,a_3]\,{}_2F_1[a_1,a_2;a_3;z]) = C[a_1,a_2,a_3]\,{}_2F_1[a_1,a_2;a_3;z]; \quad \forall F \in X$$

We have the following theorem:

Theorem 2. *Let $\mathcal{A}_i^{\alpha_i}, i = 1,2,3$ and $\mathcal{I}$ defined as in Definition (1), then*

$$\mathcal{A}_3^{-1} = \frac{a_1}{a_3 - 1}\mathcal{A}_1 + \frac{a_3 - a_1 - 1}{a_3 - 1}\mathcal{I}; \quad a_3 \neq 1 \tag{8}$$

$$\mathcal{A}_2^{-1} = \frac{a_1(z-1)}{a_2 - a_3}\mathcal{A}_1 + \frac{a_1 + a_2 - a_3}{a_2 - a_3}\mathcal{I}; \quad a_2 \neq a_3 \tag{9}$$

$$\mathcal{A}_1^{-1} = \frac{a_1(z-1)}{a_1 - a_3}\mathcal{A}_1 + \frac{2a_1 + (a_2 - a_1)z - a_3}{a_1 - a_3}\mathcal{I}; \quad a_1 \neq a_3 \tag{10}$$

$$\mathcal{A}_2 = \frac{a_1}{a_2}\mathcal{A}_1 + \frac{a_2 - a_1}{a_2}\mathcal{I}; \quad a_2 \neq 0 \tag{11}$$

$$\mathcal{A}_3 = \frac{a_1 a_3(z-1)}{(a_1 - a_3)(a_3 - a_2)z}\mathcal{A}_1 - \frac{a_3((a_3 - a_2)z - a_1)}{(a_1 - a_3)(a_3 - a_2)z}\mathcal{I}; \quad a_1 \neq a_3, \ a_2 \neq a_3 \text{ and } z \neq 0 \tag{12}$$

Proof 1. To prove (8), from Eq. (45) of [15], and with $\alpha_1 = \alpha_2 = \alpha_3 = 0$, one has

$$\left[\mathcal{I} - \mathcal{A}_1^{-1} - \frac{a_2}{a_3}z\mathcal{A}_2\mathcal{A}_3\right]\ {}_2F_1[a_1, a_2; a_3; z] = 0$$

that is

$$\mathcal{A}_1 = \mathcal{I} + \frac{a_2}{a_3}z\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3 \tag{13}$$