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REVIEW PAPER

Some quantum optical states as realizations of Lie groups

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Abstract We start with the Heisenberg–Weyl algebra and after the definitions of the Fock states we give the definition of the coherent state of this group. This is followed by the exposition of the $SU(2)$ and $SU(1,1)$ algebras and their coherent states. From there we go on describing the binomial state and its extensions as realizations of the $SU(2)$ group. This is followed by considering the negative binomial states, and some squeezed states as realizations of the $SU(1,1)$ group. Generation schemes based on physical systems are mentioned for some of these states.

1. Introduction

With the advances in the field of quantum optics which began with the 60s, group theory started to infiltrate in this branch. Groups involving simple Lie algebras such as $SU(2)$ and $SU(1,1)$ and their simple generalizations have been used to study different aspects in quantum optics. However, the use of the theory of groups in quantum mechanics started with the early days of that theory. Weyl's book that was first published in German in 1928 [1] is a standing witness on this. Wider dimensions in various branches of physics benefited greatly from the use of the group theory.

Some states used in the field of quantum optics as realizations of the $SU(2)$ or $SU(1,1)$ groups are reviewed. We start by some preliminaries about the annihilation and creation operators and the number operators which constitute the corner stones of the Hesenbeg-Weyl algebra, then their eigenstates and their coherent states are defined. The familiar algebras of the $SU(2)$ and $SU(1,1)$ are introduced. Then some quantum states which are realizations of the $SU(2)$ are reviewed in Section 3. Section 4 is devoted to states as realizations of $SU(1,1)$ group. Some comments are given about the generations of some of these states through physical processes.

2. Preliminaries

2.1. The harmonic oscillator

In the study of the harmonic oscillator, the following operators are introduced: the annihilation operator $\hat{a}$ the creation operator $\hat{a}^\dagger$ and the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$. They satisfy the commutation relations

$$[a, a^\dagger] = I, \quad [n, a^\dagger] = a^\dagger, \quad [n, a] = -a. \tag{1}$$

The eigen-states $|n\rangle$ of the number operator $\hat{n}$ are called Fock states or number states. They satisfy

$$\hat{n}|n\rangle = n|n\rangle. \tag{2}$$

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The non-negative integer n can be looked upon as the number of particles in the state. When $n = 0$ we call $|0\rangle$ the vacuum state with no particles present.

The operations of a and $a^\dagger$ on $|n\rangle$ are given by

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{3}$$

The states $\{|n\rangle\}$ form a complete set and resolve the unity

$$\sum_n |n\rangle\langle n| = I. \tag{4}$$

2.1.1. Coherent states

The coherent state $|\alpha\rangle$ can be looked upon as an eigenstate of the operator a such that

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{5}$$

Also, it can be produced by applying the Glauber displacement operator which is a unitary operator on the vacuum state $|0\rangle$ [2,3].

$$|\alpha\rangle = D(\alpha)|0\rangle = \exp(\alpha a^\dagger - \alpha^* a)|0\rangle.$$

This is the coherent state of the Heisenberg–Weyl group [4,5].

This state, which is a superposition of infinite series of the Fock states with their distribution being Poissonian. It is given by its expansion in the number state as

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n|n\rangle, \quad C_n = e^{-\frac{1}{2}|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}. \tag{6}$$

This state describes to a great deal the laser field where the phase is fixed while the number is not. The sates $\{|\alpha\rangle\}$ are overcomplete and they satisfy $\int |\alpha\rangle\langle\alpha|\frac{d^2\alpha}{\pi} = I$.

2.2. The angular momentum (SU(2) group)

The angular momentum defined as $\hat{r}\times\hat{p}$ as well as the spin, are described by the three operators J_x, J_y, and J_z which satisfy the commutation relations (we take $\hbar = 1$)

$$[J_x, J_y] = iJ_z, \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y, \tag{7}$$

with

$$J^2 = J_x^2 + J_y^2 + J_z^2,$$

which commutes with each component. Raising and lowering operators are introduced through the relations

$$J_\pm = J_x \pm iJ_y.$$

Hence the commutation relations (7) become

$$[J_z, J_\pm] = \pm J_\pm \quad \text{and} \quad [J_+, J_-] = 2J_z. \tag{8}$$

The simultaneous eignestates of the operators J_z and J^2 denoted by $|j,m\rangle$ are given from [5,6]

$$J^2|j,m\rangle = j(j+1)|j,m\rangle \quad \text{and} \quad J_z|j,m\rangle = m|j,m\rangle, \tag{9}$$

with $|m| \leqslant j$, j half integers.

The operations of J_+ and J_- on $|j,m\rangle$ are given by

$$\begin{aligned} J_+|j,m\rangle &= \sqrt{(j-m)(j+m+1)}\ |j,m+1\rangle \\ J_-|j,m\rangle &= \sqrt{(j+m)(j-m+1)}\ |j,m-1\rangle. \end{aligned} \tag{10}$$

The operators J_α are the generators of the group $SU(2)$. The angular momentum coherent state is defined by the action of the rotation operator

$$\widehat{R}(\theta,\phi) = \exp\left[\frac{1}{2}\theta\left(e^{-i\phi}J_+ - e^{i\phi}J_-\right)\right], \tag{11}$$

on the state $|j,-j\rangle$.

The angular momentum coherent state $|\theta,\phi\rangle$ is given by

$$|\theta,\phi\rangle = \widehat{R}(\theta,\phi)|j,-j\rangle = \left(\cos\frac{1}{2}\theta\right)^{2j}\sum_{m=-j}^{j}\sqrt{\binom{2j}{j+m}}\left(\tan\frac{1}{2}\theta e^{-i\phi}\right)^{j+m}|j,m\rangle. \tag{12}$$

They resolve the identity operator on the space with total angular momentum j as follows

$$\frac{2j+1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\sin\theta\, d\theta\, d\phi|\theta,\phi\rangle\langle\theta,\phi| = I. \tag{13}$$

2.3. The SU(1,1) group

The notion of coherent states can be extended to any set of operators obeying a Lie algebra. The $SU(1,1)$ is the simplest non-abelian noncompact Lie group with a simple Lie algebra (For a comprehensive review we may refer to [6] and the recent review book [7]).

The $SU(1,1)$ algebra is spanned by the three operators K_1, K_2, K_3 which satisfy the commutation meatiness

$$[K_1, K_2] = -iK_3, \quad [K_2, K_3] = iK_1, \quad [K_3, K_1] = iK_2.$$

By using the operators $K_\pm = K_1 \pm iK_2$, hence

$$[K_3, K_\pm] = \pm K_\pm \quad \text{and} \quad [K_+, K_-] = -2K_3. \tag{14}$$

The Casimir operator $C^2 = K_3^2 - K_1^2 - K_2^2$ has the value $C^2 = k(k-1)I$ for any irreducible representation. Thus, representation is determined by the parameter k which is called the Bargmann number. The corresponding Hilbert space is spanned by the complete orthonormal basis $\{|k,n\rangle\}$ which are the eigenstates of C^2 and K_3, such that

$$\langle k,n|k,m\rangle = \delta_{nm} \quad \text{and} \quad \sum_{n=0}^{\infty}|k,n\rangle\langle k,n| = I.$$

The operations of the operators $K_\pm$ and K_3 on $|k,n\rangle$ are given by [5]

$$\begin{aligned} K_+|k,n\rangle &= \sqrt{(n+1)(2k+n)}\ |k,n+1\rangle \\ K_-|k,n\rangle &= \sqrt{n(2k+n-1)}\ |k,n-1\rangle \\ K_3|k,n\rangle &= (k+n)|k,n\rangle \end{aligned} \quad . \tag{15}$$

The ground state $|k,0\rangle$ satisfies $K_-|k,0\rangle = 0$ while

$$|k,m\rangle = \sqrt{\frac{\Gamma(2k)}{n!\Gamma(2k+m)}}K_+^m|k,0\rangle.$$

There are two sets of coherent states related to the $SU(1,1)$ group namely:

(i) The Perelomov coherent states.By applying the unitary operator

$$D_{Per}(\xi) = \exp(\xi K_+ - \xi^* K_-),$$

on the ground state $|k,0\rangle$ to get [2]