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Fubini theorem for multiparameter stable process

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Abstract We prove stochastic Fubini theorem for general stable measure which will be used to develop some identities in law for functionals of one and two-parameter stable processes. This result is subsequently used to establish the integration by parts formula for stable sheet.

1. Introduction

Stochastic Fubini theorem for double Wiener integrals was first proved by Donati-Martin and Yor [4] and then developed further by Yor and other researchers. See [3] and the references therein. Subsequently, this theorem was applied to establish some identities in law for some quadratic functionals of Brownian motion. Among these identities in law there is one similar to the integration by parts formula which allowed some interesting extensions of the famous Ciesielski-Taylor identity. A simple explanation of the Ciesielski-Taylor identity is presented in the paper [10]. In view of this, it is natural to ask if it is possible to developed this for other processes. Because of their generality, Lévy processes, in particular stable processes, have been the object of intense research activity in recent years (see e.g. [1,2] and [9]). In this regard it would be of interest to have a stochastic Fubini theorem for such processes. The first adequate extension of Stochastic Fubini theorem to symmetric stable process and the related results was established by Donati-Martin et al. [5].

Generalization of some well-known results for stochastic processes indexed by a single parameter to those indexed by two parameters has attracted considerable interest recently. In general, processes parametrized by two parameters can provide more flexibility in their applications in modelling physical phenomena. Of particular interest, for which several generalizations have been established, are the Brownian sheet and bivariate Brownian bridge. For example, as a consequences of Stochastic Fubini theorem for general Gaussian measures, the authors in [3] have obtained some identities in law, integration by parts formula and the law of a double stochastic integral for such processes. In the same context the authors in [7] have established new identities in law for quadratic functionals of conditioned bivariate Gaussian processes. In particular, their results provide a two-parameter generalization of a celebrated identity in law, involving the path variance of a Brownian bridge, due to Watson [12]. We will see how this kind of identities can be naturally extended to stable processes.

In Section 2, as a first step we establish stochastic Fubini theorem for general Stable measure. This brings us, first, to an identity in law of functionals of one parameter time changed stable process. In fact we extent the well-known identity

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in law involving quadratic functionals of the Brownian bridge (for more details see [11]), which corresponds to 2-stable process, to general α-stable process. As a second consequence we produce, using minor additional technicalities, the same results for the well-known symmetric α-stable sheet $\{X^\alpha(t_1,t_2), (t_1,t_2) \in [0,1]^2\}$, which may be described as follows:

Let $\mathcal{I}$ be the class of all sets in $[0,1]^2$ of the type $\times_{i=1}^{2}(s_i,t_i]$, $s_i,t_i \in [0,1]$. For a given a function $f:[0,1]^2 \to \mathbb{R}$, the increment $f(I)$ of f over the set $I \in \mathcal{I}$ is defined by

$$f\left(\times_{i=1}^{2}(s_i,t_i]\right) = f(t_1,t_2) - f(t_1,s_2) - f(s_1,t_2) + f(s_1,s_2).$$

For $\alpha \in (0,2]\backslash\{1\}$, X^α is a stochastic process taking values in $\mathbb{R}$ defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ such that:

(i) For any $k \in \mathbb{N}$ and any choice of disjoint sets $I_j \in \mathcal{I}$, $j \in \{1,\ldots,k\}$ the increments $X(I_j)$ are independents.

(ii) For any $I \in \mathcal{I}$ and $u \in \mathbb{R}$

$$\mathbb{E}\exp(iuX(I)) = \exp(-\lambda(I)|u|^\alpha), \tag{1}$$

where $\lambda(I)$ is the Lebesgue measure of I.

It is well known that X^α belongs to the space $D([0,1]^2,\mathbb{R})$ of functions Z from $[0,1]^2$ into $\mathbb{R}$ vanishing at the boundary and satisfying

$$\lim_{(t_1,t_2)\leqslant(s_1,s_2),(s_1,s_2)\to(t_1,t_2)} Z(s_1,s_2) = Z(t_1,t_2),$$

where $\leqslant$ denotes the natural partial ordering in $[0,1]^2$.

It should be noted that our results for the stable sheet are actually a continuation of those established by Peccati and Yor [7] for the Brownian sheet.

Section 3 is devoted to the integration by parts formula established first in [11] for the Brownian motion. Since then several extensions have been made to various processes. Namely the first one, for the one parameter stable process, was given in [5] whereas the second one, for Brownian sheet, was made in [3]. We are going here to show this formula for stable sheet. Our proof is based on the main result of Section 2 and time reversal stochastic integral with respect to stable process.

Let us fix some notations to be used throughout the paper: $X \overset{d}{=} Y$ means that the random variables X and Y have the same distribution. T_γ is a one-sided stable random variable with exponent γ if $\mathbb{E}(\exp(-uT_\gamma)) = \exp(-u^\gamma)$, for $u \geqslant 0$.

2. Some identities in law between some Lévy functionals

The starting point of this study is Fubini theorem for Stable measures. Let $(A,\mathcal{A},\mu)$ and $(B,\mathcal{B},\nu)$ be two measurable spaces, with μ and ν denoting positive and σ-finite measures.

Let $\left\{X^\alpha_\mu(h) : h \in L^\alpha(A,\mathcal{A},\mu)\right\}$ and $\left\{X^\beta_\nu(k) : k \in L^\beta(B,\mathcal{B},\nu)\right\}$ be two independent stable symmetric processes, with $\alpha,\beta \in (0,2]\backslash\{1\}$, indexed respectively by functions in $L^\alpha(A,\mathcal{A},\mu)$ and $L^\beta(B,\mathcal{B},\nu)$, that is, for any $u \in \mathbb{R}$, $h \in L^\alpha(A,\mathcal{A},\mu)$ and $k \in L^\beta(B,\mathcal{B},\nu)$, we have

$$\mathbb{E}\left[\exp\left\{iuX^\alpha_\mu(h)\right\}\right] = \exp\left\{-\int_A |uh(a)|^\alpha \mu(da)\right\},$$

and

$$\mathbb{E}\left[\exp\left\{iuX^\beta_\nu(k)\right\}\right] = \exp\left\{-\int_B |uk(b)|^\beta \nu(db)\right\}.$$

Here we give some examples:

Let $\left\{X^\alpha_t, t \in [0,1]\right\}$ be a symmetric stable process with index α, that is a Lévy process such that for any $t \in [0,1]$ and $u \in \mathbb{R}$ its characteristic function is defined by

$$\mathbb{E}\left[\exp\left\{iuX^\alpha_t\right\}\right] = \exp(-t|u|^\alpha).$$

1. For $(A,\mathcal{A},\mu) = ([0,1],\mathcal{B}([0,1]),dt)$, then $X^\alpha_\mu(h)$ has a stochastic integral representation
$$X^\alpha_\mu(h) \overset{d}{=} \int_0^1 h(s)dX^\alpha_s.$$
2. For $(A,\mathcal{A},\mu) = ([0,1],\mathcal{B}([0,1]),\eta(dt))$, where η denotes a positive and σ-finite measure such that $\eta(\{0\}) = 0$, we have
$$X^\alpha_\mu(h) \overset{d}{=} \int_0^1 h(s)dX^\alpha_{\eta[0,s]}.$$
3. For $(A,\mathcal{A},\mu) = ([0,1]^2,\mathcal{B}([0,1]^2),dt\,ds)$, $X^\alpha_\mu(h)$ has the law representation as
$$X^\alpha_\mu(h) \overset{d}{=} \int_{[0,1]^2} h(t_1,t_2)dX^\alpha(t_1,t_2).$$

We now state a fundamental identity, which holds almost surely, on which our main result Theorem (1) is based on.

$$\int_A\left(\int_B \phi(a,b)X^\beta_\nu(db)\right)X^\alpha_\mu(da) = \int_B\left(\int_A \phi(a,b)X^\alpha_\mu(da)\right)X^\beta_\nu(db), \tag{2}$$

for any $\phi : A \times B \to \mathbb{R}$, $\mathcal{A}\otimes\mathcal{B}$-measurable function such that

$$\int_A\left|\int_B |\phi(a,b)|^\beta \nu(db)\right|^{\alpha/\beta}\mu(da) < +\infty,$$

and

$$\int_B\left|\int_A |\phi(a,b)|^\alpha \mu(da)\right|^{\beta/\alpha}\nu(db) < +\infty.$$

The main result in this section, which is fundamental for the rest of the development, is as follows:

Theorem 1. *Consider for $\alpha,\beta \in (0,2]\backslash\{1\}$ the random variables*

$$Y_{\beta,\alpha} = \int_A\left|\int_B \phi(a,b)X^\beta_\nu(db)\right|^\alpha \mu(da)$$

and

$$Y_{\alpha,\beta} = \int_B\left|\int_A \phi(a,b)X^\alpha_\mu(da)\right|^\beta \nu(db).$$

Then the following identity holds

$$\left(Y_{\beta,\alpha}\right)^{1/\gamma}T_\gamma \overset{d}{=} Y_{\alpha,\beta}, \tag{3}$$

where $\gamma = \alpha/\beta$ and T_γ is a one-sided stable random variable with exponent γ, which is assumed to be independent of $Y_{\beta,\alpha}$.

For $\alpha = \beta$ the identity in law (3) becomes