Journal of the Egyptian Mathematical Society (2011) **19**, 78–81

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

Some results on Dunkl analysis : A survey ☆

M. Sifi *, B. Amri, J-Ph. Anker, S. Hassani, S. Mustapha

Department of Mathematics, Faculty of Sciences of Tunis, 2092 Tunis, Tunisia

Available online 9 November 2011

KEYWORDS

Dunkl operator;
Paley–Wiener theorem;
Generalized translations;
Riesz potentials;
Riesz transforms

Abstract In this article, we present a survey of some new results obtained in [2,8]. First, we give a geometric Paley–Wiener theorem for the Dunkl transform in the crystallographic case. Next we describe more precisely the support of the distribution associated to Dunkl translations in higher dimension. We also investigate the $L^p \to L^q$ boundedness properties of the Riesz potentials I_α^κ and the related fractional maximal function $M_{\kappa,\alpha}$ associated to the Dunkl transform. Finally we prove the L^p-boundedness, $(1 < p < \infty)$ of the Riesz transforms in the Dunkl setting.

1. Introduction

Dunkl theory generalizes classical Fourier analysis on $\mathbb{R}^N$. It started twenty years ago with Dunkl's seminal work [6] and was further developed by several mathematicians. See for instance the surveys [15,7] and the references cited therein.

In this setting, the Paley–Wiener theorem is known to hold for balls centered at the origin. In [9], a Paley–Wiener theorem was conjectured for convex neighborhoods of the origin, which are invariant under the underlying reflection group, and was partially proved.

Our first result in Section 2, is a proof of this conjecture in the crystallographic case, following the third approach in [9].

Generalized translations were introduced in [14] and further studied in [19,16,17]. Apart from their abstract definition, we lack precise information, in particular about their integral representation

$$(\tau_x f)(y) = \int_{\mathbb{R}^N} f(z)\, d\gamma_{x,y}(z),$$

which was conjectured in [14] and established in few cases, for instance in dimension $N = 1$ or when f is radial.

Our second result in Section 2 deals with the support of the distribution $\gamma_{x,y}$ in higher dimension, that we determine rather precisely in the crystallographic case.

For $0 < \alpha < 2\gamma_\kappa + d$, the Riesz potential $I_\alpha^\kappa f$ is defined on $\mathcal{S}(\mathbb{R}^d)$ (the class of Schwartz functions) by (see [18])

$$I_\alpha^\kappa f(x) = (d_\kappa^\alpha)^{-1} \int_{\mathbb{R}^d} \frac{\tau_y f(x)}{\|y\|^{2\gamma_\kappa + d - \alpha}} h_\kappa^2(y)\, dy, \qquad (1)$$

where

$$d_\kappa^\alpha = 2^{-\gamma_\kappa - d/2 + \alpha} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\gamma_\kappa + \frac{d-\alpha}{2})}.$$

It is easy to see that the Riesz potentials operate on the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, as integral operators, and it is natural

☆ Author partially supported by DGRST project 04/UR/15-02 and CMCU program 10G 1503.
* Corresponding author.
E-mail address: mohamed.sifi@fst.rnu.tn (M. Sifi).

1110-256X

Peer review under responsibility of Egyptian Mathematical Society.
doi:10.1016/j.joems.2011.09.001

Production and hosting by Elsevier

to inquire about their action on the spaces $L^p(\mathbb{R}^d, h_\kappa^2)$. The main problem can be formulated as follows. Given $\alpha \in]0, 2\gamma_\kappa + d[$ for what pair (p,q) is it possible to extend (1) to a bounded operator from $L^p(\mathbb{R}^d, h_\kappa^2)$ to $L^q(\mathbb{R}^d, h_\kappa^2)$? That is when do we have the inequality

$$\|I_\alpha^\kappa f\|_{\kappa,q} \leqslant C\|f\|_{\kappa,p}. \tag{2}$$

The notation $\|\cdot\|_{\kappa,p}$ is used here to denote the norm of $L^p(\mathbb{R}^d, h_\kappa^2)$.

A necessary condition is given in [18]. This condition says that (2) holds only if

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\gamma_\kappa + d}. \tag{3}$$

Thangavelu and Xu proved also in [18] that the condition (3) is sufficient to ensure the boundedness of I_α^κ (save for $p = 1$ where a weak-type estimate holds) if one assumes that the reflection group G is $\mathbb{Z}_2^d$ or if f are radial functions and $p \leqslant 2$ (see [18, Theorem 4.4]).

We will show that it is possible to remove this restrictive hypothesis and prove that (3) is a sufficient condition for all reflection groups.

On $\mathbb{R}^N$, the ordinary Riesz transform R_j, $j = 1\ldots N$ is defined as the multiplier operator

$$\widehat{R_j(f)}(\xi) = -i\frac{\xi_j}{|\xi|}\widehat{f}, \quad \xi \in \mathbb{R}^N.$$

It can also be defined by the principal value of the singular integral

$$R_j(f) = \lim_{\varepsilon\to 0}\int_{|y|\geqslant\varepsilon} r_j(x-y)f(y)dy,$$

where r_j is the singular Riesz kernel given by

$$r_j(y) = \Gamma\left(\frac{N+1}{2}\right)\pi^{N+1}y_j/|y|^{N+1},$$

whose Fourier transform (in the sense of distributions) is

$$\widehat{R_j}(\xi) = -i\xi_j/|\xi|.$$

It is a classical result in harmonic analysis that the Riesz transforms are bounded on L^p for all $1 < p < \infty$.

In Dunkl setting the Riesz transforms (see [18]) are the operators $\mathcal{R}_j$, $j = 1\ldots d$ defined on $L^2(\mathbb{R}^d, h_\kappa^2)$ by

$$\mathcal{R}_j(f)(x) = c_j\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon}\tau_x(f)(-y)\frac{y_j}{|y|^m}h_\kappa^2(y), \quad x \in \mathbb{R}^d,$$

where $c_j = 2^{\gamma_k+d/2}\Gamma(\gamma_k + (d+1)/2)/\sqrt{\pi}$ and $m = 2\gamma_k + d + 1$.

The study of the L^p-boundedness of Riesz transforms for Dunkl transform on $\mathbb{R}^N$ goes back to the work of Thangavelyu and Xu [18] where they established boundedness result only in a very special case of $N = 1$. Recently Amri [1] proves this result in more general case.

As applications, we will prove the generalized Riesz and Sobolev inequalies.

2. New results about Dunkl analysis

2.1. A geometric Paley–Wiener theorem

In this subsection, we state a geometric version of the Paley–Wiener theorem, which was looked for in [9,19,10], under the assumption that G is crystallographic. The proof which was given in [2] consists merely in resuming the third approach in [9] and applying it to the convex sets considered in [3–5] instead of the convex sets considered in [11]. Recall that the second family consists of the convex hulls

$$C^\Lambda = \mathrm{co}(G\cdot\Lambda)$$

of G-orbits $G{\cdot}\Lambda$ in $\mathbb{R}^N$, while the first family consists of the polar sets

$$C_\Lambda = \{x \in \mathbb{R}^N | \langle x, g\cdot\Lambda\rangle \leqslant 1 \forall g \in G\}.$$

Before stating the geometric Paley–Wiener theorem, let us make some remarks about the sets C^Λ and C_Λ.

Firstly, they are convex, closed, G-invariant and the following inclusion holds.

$$C^\Lambda \subset |\Lambda|^2 C_\Lambda.$$

Secondly, we may always assume that $\Lambda = \Lambda_+$ belongs to the closed positive chamber $\overline{\Gamma_+}$ and, in this case, we have

$$C^\Lambda \cap \overline{\Gamma_+} = \overline{\Gamma_+} \cap \left(\Lambda - \overline{\Gamma^+}\right),$$

$$C_\Lambda \cap \overline{\Gamma_+} = \{x \in \overline{\Gamma_+} | \langle\Lambda, x\rangle \leqslant 1\}.$$

Thirdly, on one hand, every G-invariant convex subset in $\mathbb{R}^N$ is a union of sets C^Λ while, on the other hand, every G-invariant closed convex subset in $\mathbb{R}^N$ is an intersection of sets C_Λ. For instance,

$$\overline{B(0,R)} = \bigcup_{|\Lambda|=R} C^\Lambda = \bigcap_{|\Lambda|=R^{-1}} C_\Lambda.$$

Fourthly, we shall say that $\Lambda \in \overline{\Gamma_+}$ is *admissible* if the following equivalent conditions are satisfied:

(i) Λ has nonzero projections in each irreducible component of $(\mathbb{R}^N, R)$,
(ii) C^Λ is a neighborhood of the origin,
(iii) C_Λ is bounded.

In this case, we may consider the gauge

$$\chi_\Lambda(\xi) = \max_{x\in C_\Lambda}\langle x, \xi\rangle = \min\{r \in [0, +\infty) | \xi \in rC^\Lambda\}$$

on $\mathbb{R}^N$.

Theorem 1. *Assume that $\Lambda \in \overline{\Gamma_+}$ is admissible. Then the Dunkl transform is a linear isomorphism between the space of smooth functions f on $\mathbb{R}^N$ with supp $f \subset C_\Lambda$ and the space of entire functions h on $\mathbb{C}^N$ such that*

$$\sup_{\xi\in\mathbb{C}^N}(1+|\xi|)^M e^{-\chi_\Lambda(\mathrm{Im}\,\xi)}|h(\xi)| < +\infty \quad \forall M \in \mathbb{N}. \tag{4}$$

Following [9], this theorem is first proved in the trigonometric case, which explains the restriction to crystallographic groups, and next obtained in the rational case by passing to the limit. The proof of Theorem 1 in the trigonometric case is similar to the proof of the Paley–Wiener Theorem in [11,12], and actually to the initial proof of Helgason for the spherical Fourier transform on symmetric spaces of the noncompact type. The limiting procedure, as far as it is concerned, is described thoroughly in [9] and needs no further explanation.