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ORIGINAL ARTICLE

Local Lie groups and local top spaces

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Abstract In this paper a generalization of local Lie groups, using the concept of top spaces, is given and some theorems about the relation between this generalization and local Lie groups are provided.

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1. Local Lie group

The concept of Lie group was local at first. Local Lie groups defined as open subsets of Euclidean spaces and the group multiplication and inversion operators only being defined for elements sufficiently near the identity. Lie groups, in the way that we know today, are formed when the definition of manifold constructed. In this paper we generalize the concept of local Lie group, using top spaces.

The notion of generalized group was introduced by Molaei when working on constructing a geometric unified theory by using Santilli's iso theory, which is applied in mathematical physics. After that he introduced top spaces as a generalization of Lie groups by using generalized group. Let us recall the definition of generalized group and top space.

Definition 1.1 [1]. A generalized group is a non-empty set G admitting an operation called multiplication, subject to the set of rules given below:

- $(xy)z = x(yz)$, for all $x, y, z \in G$;
- For each x in G there exists a unique z in G such that $xz = zx = x$ (we denote z by $e(x)$);
- For each $x \in T$ there exists $y \in T$ such that $xy = yx = e(x)$ (we denote y by x^{-1}).

We recall that T is a topological generalized group if:

- T is a generalized group.
- T is a hausdorff topological space.
- The mappings, $m_1 : T \to T,\ m_1(x) = x^{-1}$ and $m_2 : T \times T \to T,\ m_2(x, y) = xy$ are continuous maps.

Definition 1.2 [2]. A topological generalized group $(T, \cdot)$ is called a top space if:

- T is a smooth manifold.
- The mapping $m_1 : T \to T$ is defined by $m_1(x) = x^{-1}$ and the mapping $m_2 : T \times T \to T$ is defined by $m_2(x, y) = xy$ are smooth maps.

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A top space T is called a normal top space if $e(xy) = e(x)e(y)$ for all $x, y \in T$.

Suppose that a group G and two sets Λ and I are given. If $p : I \times \Lambda \to G$ is a mapping, then $\Lambda \times G \times I$ with the product $(\lambda, g, i)(\lambda_1, g_1, i_1) = (\lambda, gp(i, \lambda_1)g_1, i_1)$ is a generalized group, which is called Rees matrix semigroup denoted by $M(G, I, \Lambda, p)$ [2].

Theorem 1.1 [2]. *If I and Λ are smooth manifolds, G is a Lie group and $p : I \times \Lambda \to G$ is a smooth map, then $M(G, I, \Lambda, p)$ is a top space.*

It is also proved in [3] that every top space with finite number of identities is diffeomorphic with $M(G, I, \Lambda, p)$, for some Lie group G and two finite subsets I and Λ. See [2–8] for more information about top spaces.

In the remaining of this section we recall the concept of local Lie groups and some basic definitions that we need in the next section. While the definition of a global Lie group is standard, the precise definition of a local Lie group varies from author to author. The following one is from [9].

Definition 1.3 [9]. A smooth manifold L is called a local Lie group if there exists

- a distinguished element $e \in L$, the identity element,
- a smooth product map $\mu : U \to L$ defined on an open subset $(\{e\} \times L) \cup (L \times \{e\}) \subset U \subset (L \times L)$,
- a smooth inversion map $i : V \to L$ defined on an open subset $e \in V \subset L$ such that $V \times i(V) \subset U$, and $i(V) \times V \subset U$,

all satisfying the following properties:

(i) Identity: $\mu(e, x) = x = \mu(x, e)$ for all $x \in L$.
(ii) Inverse: $\mu(i(x), x) = \mu(x, i(x)) = e$ for all $x \in V$.
(iii) Associativity: If $(x, y), (y, z), (\mu(x, y), z)$ and $(x, \mu(y, z))$ all belongs to U, then

$$\mu(\mu(x, y), z) = \mu(x, \mu(y, z)).$$

Example 1.1 [9]. The most basic example of a local Lie group is provided by any neighborhood $e \in N \subset G$ of the identity element in a global Lie group. Indeed we set U to be any open subset of $N \times N$ such that $(\{e\} \times N) \cup (N \times \{e\}) \subset U \subset (N \times N) \cap \mu^{-1}(N)$, and V to be any open subset of N such that $\{e\} \subset V \subset N \cap i^{-1}(N)$, and $(V \times i(V)) \cup (i(V) \times V) \subset U$. The group multiplication μ and inversion i on G then restricted to define local group multiplication and inversion maps on N.

One can define the right and left multiplication by

$$l_x(y) = \mu(x, y), \quad r_x(y) = \mu(y, x).$$

Definition 1.4 [9]. Let (L, μ, U, i, V) and $(\widetilde{L}, \widetilde{\mu}, \widetilde{U}, \widetilde{i}, \widetilde{V})$ be local Lie groups. A smooth map $\varphi : L \to \widetilde{L}$ is called a local group homomorphism if

- $\varphi \times \varphi(U) \subset \widetilde{U}$, $\varphi(V) \subset \widetilde{V}$, $\varphi(e) = \widetilde{e}$,
- $\varphi(\mu(g, h)) = \widetilde{\mu}(\varphi(g), \varphi(h))$ for $(g, h) \in U$,
- $\varphi(i(g)) = \widetilde{i}(\varphi(g))$ for $g \in V$.

A local group homomorphism is called a homeomorphism if it is one-to-one and onto with smooth inverse.

Definition 1.5 [9]. A local group is

- associative with order n if, for every $3 \leqslant m \leqslant n$, and every ordered m-tuple of group elements $(x_1, x_2, \ldots, x_m) \in L^{\times m}$, all corresponding well defined m-fold products are equal. A local group is called globally associative if it is associative with every order $n \geqslant 3$.
- globalizable if there exists a local group homeomorphism $\Phi : L \to N$ mapping L on to a neighborhood $e \in N \subset G$ of the identity of a global Lie group G.
- globally inversional if the inversion map i is defined everywhere, so that $V = L$.
- regular if, for each $x \in L$, the maps l_x and r_x are diffeomorphisms on their respective domains of definition.

Theorem 1.2 [9]. *Every inversional local Lie group is regular.*

Theorem 1.3 [9]. *A local Lie group L is globalizable if and only if it is globally associative.*

2. Local top spaces

In this section we use top space to generalize the concept of local Lie group.

Definition 2.1. A smooth manifold H is called a local top space if there exists

- a set $e(H) \subset H$, the identity elements,
- a smooth product map $\mu : U \to H$ defined on an open subset $(e(H) \times H) \cup (H \times e(H)) \subset U \subset (H \times H)$,
- a smooth inversion map $i : V \to H$ defined on an open subset $e(H) \subset V \subset H$ such that $V \times i(V) \subset U$, and $i(V) \times V \subset U$,

all satisfying the following properties:

(i) Identity: For each $x \in H$ there is a unique element $e(x)$ such that $\mu(e(x), x) = x = \mu(x, e(x))$.
(ii) Inverse: $\mu(i(x), x) = \mu(x, i(x)) = e(x)$ for all $x \in V$.
(iii) Associativity: If $(x, y), (y, z), (\mu(x, y), z)$ and $(x, \mu(y, z))$ all belongs to U, then

$$\mu(\mu(x, y), z) = \mu(x, \mu(y, z)),$$

(iv) $\mu(e(x), e(y)) = e(\mu(x, y))$, for each $x, y \in H$.
(v) $e : H \to H$ is a smooth map.

One can use the symbol (H, μ, U, i, V) for a local top space H with the functions μ, i, U and V as the above definition.

Remark 2.1

(1) Using (i), (iii) and (iv) in the above definition, we have

$$\mu(x, \mu(e(x), e(x))) = \mu(\mu(x, e(x)), e(x)) = \mu(x, e(x)) = x.$$

Hence uniqueness of the identity element implies that $\mu(e(x), e(x)) = e(x)$ and consequently $e(e(x)) = e(x)$, for every $x \in H$.