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ORIGINAL ARTICLE

Global dynamics of some systems of rational difference equations ☆

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Abstract In this paper, we study the qualitative behavior of some systems of second-order rational difference equations. More precisely, we study the equilibrium points, local asymptotic stability of equilibrium point, unstability of equilibrium points, global character of equilibrium point, periodicity behavior of positive solutions and rate of convergence of positive solutions of these systems. Some numerical examples are given to verify our theoretical results.

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1. Introduction and preliminaries

Recently there has been a great interest in studying the qualitative properties of rational difference equations. For a systematic study of rational difference equations we refer [1–15] and references therein. In Refs. [16–19] qualitative behavior of some biological models is discussed. Recently there has been a lot of interest in studying the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations.

Bajo and Liz [5] investigated the global behavior of difference equation:

$$x_{n+1} = \frac{x_{n-1}}{a + bx_{n-1}x_n},$$

for all values of real parameters a, b.

Aloqeili [6] discussed the stability properties and semi-cycle behavior of the solutions of the difference equation:

$$x_{n+1} = \frac{x_{n-1}}{a - x_{n-1}x_n}, \quad n = 0, 1, \ldots,$$

with real initial conditions and positive real number a.

Motivated by the above studies, our aim in this paper is to investigate the qualitative behavior of following systems of second-order rational difference equations:

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta - \gamma y_n y_{n-1}}, \quad y_{n+1} = \frac{\alpha_1 y_{n-1}}{\beta_1 - \gamma_1 x_n x_{n-1}}, \quad n = 0, 1, \ldots, \tag{1}$$

and

$$x_{n+1} = \frac{a y_{n-1}}{b - c x_n x_{n-1}}, \quad y_{n+1} = \frac{a_1 x_{n-1}}{b_1 - c_1 y_n y_{n-1}}, \quad n = 0, 1, \ldots, \tag{2}$$

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where the parameters $\alpha,\ \beta,\ \gamma,\ \alpha_1,\ \beta_1,\ \gamma_1,\ a,\ b,\ c,\ a_1,\ b_1,\ c_1$ and initial conditions $x_0,\ x_{-1},\ y_0,\ y_{-1}$ are positive real numbers.

Let us consider four-dimensional discrete dynamical system of the form

$$x_{n+1} = f(x_n, x_{n-1}, y_n, y_{n-1}), \quad (3)$$
$$y_{n+1} = g(x_n, x_{n-1}, y_n, y_{n-1}), \quad n = 0, 1, \ldots,$$

where $f : I^2 \times J^2 \to I$ and $g : I^2 \times J^2 \to J$ are continuously differentiable functions and I, J are some intervals of real numbers. Furthermore, a solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of system (3) is uniquely determined by initial conditions $(x_i, y_i) \in I \times J$ for $i \in \{-1, 0\}$. Along with system (3) we consider the corresponding vector map $F = (f, x_n, x_{n-1}, g, y_n, y_{n-1})$. An equilibrium point of (3) is a point $(\bar{x}, \bar{y})$ that satisfies

$$\bar{x} = f(\bar{x}, \bar{x}, \bar{y}, \bar{y}),$$
$$\bar{y} = g(\bar{x}, \bar{x}, \bar{y}, \bar{y}).$$

The point $(\bar{x}, \bar{y})$ is also called a fixed point of the vector map F.

Definition 1. Let $(\bar{x}, \bar{y})$ be an equilibrium point of the system (3).

(i) An equilibrium point $(\bar{x}, \bar{y})$ is said to be stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every initial condition $(x_i, y_i),\ i \in \{-1, 0\} \left\| \sum_{i=-1}^{0} (x_i, y_i) - (\bar{x}, \bar{y}) \right\| < \delta$ implies $\|(x_n, y_n) - (\bar{x}, \bar{y})\| < \varepsilon$ for all $n > 0$, where $\|\cdot\|$ is the usual Euclidian norm in $\mathbb{R}^2$.
(ii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be unstable if it is not stable.
(iii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be asymptotically stable if there exists $\eta > 0$ such that $\left\| \sum_{i=-1}^{0} (x_i, y_i) - (\bar{x}, \bar{y}) \right\| < \eta$ and $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.
(iv) An equilibrium point $(\bar{x}, \bar{y})$ is called global attractor if $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.
(v) An equilibrium point $(\bar{x}, \bar{y})$ is called asymptotic global attractor if it is a global attractor and stable.

Definition 2. Let $(\bar{x}, \bar{y})$ be an equilibrium point of the map

$$F = (f, x_n, x_{n-1}, g, y_n, y_{n-1}),$$

where f and g are continuously differentiable functions at $(\bar{x}, \bar{y})$. The linearized system of (3) about the equilibrium point $(\bar{x}, \bar{y})$ is

$$X_{n+1} = F(X_n) = F_J X_n,$$

where $X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix}$ and F_J is the Jacobian matrix of the system (3) about the equilibrium point $(\bar{x}, \bar{y})$.

Lemma 1 [2]. *For the system $X_{n+1} = F(X_n)$, $n = 0, 1, \ldots$ of difference equations such let $\overline{X}$ be a fixed point of F. If all eigenvalues of the Jacobian matrix J_F about $\overline{X}$ lie inside an open unit disk $|\lambda| < 1$, then $\overline{X}$ is locally asymptotically stable. If one of them has norm greater than one, then $\overline{X}$ is unstable.*

Lemma 2 [3]. *Assume that $X_{n+1} = F(X_n)$, $n = 0, 1, \ldots$ is a system of difference equations and $\overline{X}$ is the equilibrium point of this system. The characteristic polynomial of this system about the equilibrium point $\overline{X}$ is $P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$, with real coefficients and $a_0 > 0$. Then all roots of the polynomial $P(\lambda)$ lies inside the open unit disk $|\lambda|$ if and only if $\Delta_k > 0$ for $k = 0, 1, \ldots$, where Δ_k is the principal minor of order k of the $n \times n$ matrix*

$$\Delta_n = \begin{pmatrix} a_1 & a_3 & a_5 & \ldots & 0 \\ a_0 & a_2 & a_4 & \ldots & 0 \\ 0 & a_1 & a_3 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \ldots & a_n \end{pmatrix}. \quad (4)$$

The following result gives the rate of convergence of solution of a system of difference equations

$$X_{n+1} = (A + B(n))X_n, \quad (5)$$

where X_n is an m-dimensional vector, $A \in C^{m\times m}$ is a constant matrix, and $B : \mathbb{Z}^+ \to C^{m\times m}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \quad (6)$$

as $n \to \infty$, where $\|\cdot\|$ denotes any matrix norm which is associated with the vector norm

$$\|(x, y)\| = \sqrt{x^2 + y^2}.$$

Proposition 1 (*Perron's Theorem [20]*). *Suppose that condition (6) holds. If X_n is a solution of (5), then either $X_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} (\|X_n\|)^{1/n} \quad (7)$$

exists and is equal to the modulus of one the eigenvalues of matrix A.

Proposition 2 [20]. *Suppose that condition (6) holds. If X_n is a solution of (5), then either $X_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \frac{\|X_{n+1}\|}{\|X_n\|} \quad (8)$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

2. On the system $x_{n+1} = \frac{\alpha x_{n-1}}{\beta - \gamma y_n y_{n-1}},\ y_{n+1} = \frac{\alpha_1 y_{n-1}}{\beta_1 - \gamma_1 x_n x_{n-1}}$

In this section, we shall investigate the qualitative behavior of the system (1). Let $(\bar{x}, \bar{y})$ be an equilibrium point of system (1), then for $\beta > \alpha$ and $\beta_1 > \alpha_1$ system (1) has following two equilibrium points $P_0 = (0, 0)$, $P_1 = \left(\sqrt{\frac{\beta_1 - \alpha_1}{\gamma_1}}, \sqrt{\frac{\beta - \alpha}{\gamma}}\right)$.

To construct corresponding linearized form of the system (1) we consider the following transformation:

$$(x_n, x_{n-1}, y_n, y_{n-1}) \mapsto (f, f_1, g, g_1), \quad (9)$$

where $f = \frac{\alpha x_{n-1}}{\beta - \gamma y_n y_{n-1}}$, $f_1 = x_n$, $g = \frac{\alpha_1 y_{n-1}}{\beta_1 - \gamma_1 x_n x_{n-1}}$, $g_1 = y_n$. The Jacobian matrix about the fixed point $(\bar{x}, \bar{y})$ under the transformation (14) is given by

$$F_J(\bar{x}, \bar{y}) = \begin{pmatrix} 0 & \delta_1 & \delta_2 & \delta_2 \\ 1 & 0 & 0 & 0 \\ \delta_3 & \delta_3 & 0 & \delta_4 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$