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ORIGINAL ARTICLE

# Asymptotic behavior of an anti-competitive system of second-order difference equations

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**Abstract** In this paper, we study the boundedness and persistence, existence and uniqueness of positive equilibrium, local and global behavior of positive equilibrium point, and rate of convergence of positive solutions of following system of rational difference equations

$$x_{n+1} = \frac{\alpha_1 + \beta_1 y_{n-1}}{a_1 + b_1 x_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_{n-1}}{a_2 + b_2 y_n},$$

where the parameters $\alpha_i,\ \beta_i,\ a_i,\ b_i$ for $i \in \{1, 2\}$ and initial conditions $x_0,\ x_{-1},\ y_0,\ y_{-1}$ are positive real numbers. Some numerical examples are given to verify our theoretical results.

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## 1. Introduction

Systems of nonlinear difference equations of higher-order are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of systems differential and delay differential equations which model various phenomena in biology, ecology, physiology, physics, engineering and economics. For applications and basic theory of rational difference equations we refer to [1–3]. It is very interesting to investigate the behavior of solutions of a system of nonlinear difference equations and to discuss the local asymptotic stability of their equilibrium points. Competitive and anti-competitive systems of rational difference equations are very important in population dynamics. The theory of these systems has remarkable applications in biological sciences.

Gibbons et al. [4] investigated the qualitative behavior of the following second-order rational difference equations

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{\gamma + x_n}.$$

Recently, Din *et al.* [5] studied the qualitative behavior of the following competitive system of rational difference equations

$$x_{n+1} = \frac{\alpha_1 + \beta_1 x_{n-1}}{a_1 + b_1 y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 y_{n-1}}{a_2 + b_2 x_n}.$$

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Motivated by above study, our aim in this paper was to investigate the qualitative behavior of positive solutions of following second-order system of rational difference equations:

$$x_{n+1} = \frac{\alpha_1 + \beta_1 y_{n-1}}{a_1 + b_1 x_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_{n-1}}{a_2 + b_2 y_n}, \tag{1}$$

where the parameters $\alpha_i, \beta_i, a_i, b_i$ for $i \in \{1,2\}$ and initial conditions $x_0, x_{-1}, y_0, y_{-1}$ are positive real numbers.

More precisely, we investigate the boundedness character, persistence, existence and uniqueness of positive steady-state, local asymptotic stability and global behavior of unique positive equilibrium point, and rate of convergence of positive solutions of system (1) which converge to its unique positive equilibrium point.

## 2. Boundedness and persistence

In the following theorem we show the boundedness and persistence of the positive solutions of system (1). We refer to [5–8] for similar methods to prove boundedness and persistence.

**Theorem 1.** *Assume that $\beta_1\beta_2 < a_1a_2$, then every positive solution $\{(x_n, y_n)\}$ of system (1) is bounded and persists.*

**Proof.** For any positive solution $\{(x_n, y_n)\}$ of system (1), one has

$$x_{n+1} \leqslant A_1 + B_1 y_{n-1}, \quad y_{n+1} \leqslant A_2 + B_2 x_{n-1}, \quad n = 0, 1, 2, \ldots, \tag{2}$$

where $A_i = \frac{\alpha_i}{a_i}$ and $B_i = \frac{\beta_i}{a_i}$ for $i \in \{1,2\}$. Consider the following system of difference equations

$$u_{n+1} = A_1 + B_1 v_{n-1}, \quad v_{n+1} = A_2 + B_2 u_{n-1}, \quad n = 0, 1, 2, \ldots. \tag{3}$$

Solution of system (3) is given by

$$\begin{aligned} u_n = {} & \frac{A_1 + A_2B_1}{1 - B_1B_2} - \frac{\sqrt[4]{B_1}c_1\left(\left(-\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - \left(\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - i\left(i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + i\left(-i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n\right)}{4B_2^{3/4}} \\ & + \frac{\sqrt{B_1}c_2\left(\left(-\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + \left(\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - \left(-i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - \left(i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n\right)}{4\sqrt{B_2}} \\ & + \frac{1}{4}c_3\left(\left(-\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + \left(\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + \left(-i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + \left(i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n\right) \\ & - \frac{c_4\left(\left(-\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - \left(\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - i\left(-i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + i\left(i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n\right)}{4\sqrt[4]{B_1}\sqrt[4]{B_2}}, \quad n = 1, 2, \cdots, \end{aligned}$$

$$\begin{aligned} v_n = {} & \frac{A_1B_2 + A_2}{1 - B_1B_2} - \frac{\sqrt[4]{B_2}c_3\left(\left(-\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - \left(\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - i\left(i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + i\left(-i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n\right)}{4B_1^{3/4}} \\ & + \frac{\sqrt{B_2}c_4\left(\left(-\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + \left(\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - \left(-i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - \left(i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n\right)}{4\sqrt{B_1}} \\ & + \frac{1}{4}c_2\left(\left(-\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + \left(\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + \left(-i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + \left(i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n\right) \\ & - \frac{c_1\left(\left(-\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - \left(\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - i\left(-i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + i\left(i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n\right)}{4\sqrt[4]{B_1}\sqrt[4]{B_2}}, \quad n = 1, 2, \ldots, \end{aligned}$$

where $c_i$ for $i \in \{1,2,3,4\}$ depend upon initial conditions $u_{-1}, u_0, v_{-1}, v_0$. Assume that $\beta_1\beta_2 < a_1a_2$, then the sequences $\{u_n\}$ and $\{v_n\}$ are bounded, which implies that the sequences $\{x_n\}$ and $\{y_n\}$ are also bounded. Suppose that $u_{-1} = x_{-1}$, $u_0 = x_0$, $v_{-1} = y_{-1}$ and $v_0 = y_0$, then by comparison we have

$$x_n \leqslant \frac{\alpha_1 a_2 + \alpha_2\beta_1}{a_1a_2 - \beta_1\beta_2} = U_1, \quad y_n \leqslant \frac{\alpha_2 a_1 + \alpha_1\beta_2}{a_1a_2 - \beta_1\beta_2} = U_2, \quad n = 1, 2, \ldots. \tag{4}$$

Furthermore, from system (1) and (4) we obtain that

$$x_{n+1} \geqslant \frac{\alpha_1}{a_1 + b_1x_n} \geqslant \frac{\alpha_1(a_1a_2 - \beta_1\beta_2)}{a_1(a_1a_2 - \beta_1\beta_2) + b_1(\alpha_1a_2 + \alpha_2\beta_1)} = L_1, \tag{5}$$

$$y_{n+1} \geqslant \frac{\alpha_2}{a_2 + b_2y_n} \geqslant \frac{\alpha_2(a_1a_2 - \beta_1\beta_2)}{a_2(a_1a_2 - \beta_1\beta_2) + b_2(\alpha_2a_1 + \alpha_1\beta_2)} = L_2. \tag{6}$$

From (4)–(6), it follows that

$$L_1 \leqslant x_n \leqslant U_1, \quad L_2 \leqslant y_n \leqslant U_2, \quad n = 1, 2, \ldots.$$

Hence, theorem is proved. □

**Lemma 1.** *Let $\{(x_n, y_n)\}$ be a positive solution of system (1). Then, $[L_1, U_1] \times [L_2, U_2]$ is invariant set for system (1).*

**Proof.** The proof follows by induction. □

## 3. Stability analysis

To construct corresponding linearized form of system (1) we consider the following transformation:

$$(x_n, y_n, x_{n-1}, y_{n-1}) \mapsto (f, g, f_1, g_1), \tag{7}$$

where $f = x_{n+1}, g = y_{n+1}, f_1 = x_n$ and $g_1 = y_n$. The linearized system of (1) about $(\bar{x}, \bar{y})$ is given by

$$Z_{n+1} = F_J(\bar{x}, \bar{y})Z_n,$$