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ORIGINAL ARTICLE

Lacunary ideal convergence of multiple sequences☆

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Abstract An ideal I is a family of subsets of $\mathbb{N} \times \mathbb{N}$ which is closed under taking finite unions and subsets of its elements. In this article, the concept of lacunary ideal convergence of double sequences has been introduced. Also the relation between lacunary ideal convergent and lacunary Cauchy double sequences has been established. Furthermore, the notions of lacunary ideal limit point and lacunary ideal cluster points have been introduced and find the relation between these two notions. Finally, we have studied the properties such as solidity, monotonic.

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1. Introduction

The notion of statistical convergence for sequences of real numbers has been introduced by Steinhaus [1] and Fast [2] independently. Mursaleen and Edely [3] extended the above idea from single to double sequences of scalars and established relations between statistical convergence and strongly Cesáro summable double sequences. The notion of I-convergence was studied at initial stage by Kostyrko et al. [4]. Kostyrko et al. [5] gave some of basic properties of I-convergence and dealt with extremal I-limit points. Later on it was studied by Šalát et al. [6], Hazarika and Savaş [7], Tripathy and Hazarika [8] and many others. In [9], Tripathy and Tripathy introduced the notion of ideal convergent double sequences. Fridy and Orhan [10] introduced the concept of lacunary statistical convergence. Some work on lacunary statistical convergence can be found in [11–16]. The notion of lacunary ideal convergence of real sequences was introduced in [17,18]. Hazarika [19–21] introduced the lacunary ideal convergent sequences of fuzzy real numbers and studied some basic properties of this notion. Hazarika [22] introduced the notion of lacunary ideal convergent double sequences of fuzzy real numbers. Bakery and Mohammed [23] introduced lacunary mean ideal convergence in generalized random n-normed spaces.

A family of sets $I \subseteq 2^{\mathbb{N}}$ (power sets of $\mathbb{N}$) is said to be an *ideal* if I is additive i.e. $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e. $A \in I, B \subseteq A \Rightarrow B \in I$. A non-empty family of sets $F \subset 2^{\mathbb{N}}$ is a filter on $\mathbb{N}$ if and only if $\phi \notin F, A \cap B \in F$ for each $A, B \in F$, and any superset of an element of F is in F. An ideal I is called *non-trivial* if $I \neq \phi$ and $\mathbb{N} \notin I$. Clearly I is a non-trivial ideal if and only if $F = F(I) = \{\mathbb{N} - A : A \in I\}$ is a filter

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in $\mathbb{N}$, called the filter associated with the ideal I. A non-trivial ideal I is called *admissible* if and only if $\{\{n\} : n \in \mathbb{N}\} \subset I$. A non-trivial ideal I is maximal if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset (for details on ideals see [4]).

A *lacunary sequence* is an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$. The intervals determined by θ will be defined by $J_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be defined by ϕ_r (for details on lacunary sequence see [24]).

2. Definitions and preliminaries

We denote w is the space of all sequences.

Definition 1 [4]. A sequence $(x_k) \in w$ is said to be *I*-convergent to the number L if for every $\varepsilon > 0$, $\{k \in \mathbb{N} : |x_k - L| \geqslant \varepsilon\} \in I$. We write $I - \lim x_k = L$.

Definition 2 [25]. A sequence $(x_k) \in w$ is said to be *I*-null if $L = 0$. We write $I - \lim x_k = 0$.

Definition 3 [25]. Let I be an admissible ideal of $\mathbb{N}$. A sequence $(x_k) \in w$ is said to be *I-Cauchy* if for every $\varepsilon > 0$ there exists a number $m = m(\varepsilon)$ such that $\{k \in \mathbb{N} : |x_k - x_m| \geqslant \varepsilon\} \in I$.

Definition 4 [17,18]. Let $\theta = (k_r)$ be lacunary sequence. Then a sequence (x_k) is said to be *lacunary I-convergent* if for every $\varepsilon > 0$ such that

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in J_r}|x_k - L| \geqslant \varepsilon\right\} \in I.$$

We write $I_\theta - \lim x_k = L$.

Definition 5 [17,18]. Let $\theta = (k_r)$ be lacunary sequence. Then a sequence (x_k) is said to be *lacunary I-null* if for every $\varepsilon > 0$ such that

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in J_r}|x_k| \geqslant \varepsilon\right\} \in I.$$

We write $I_\theta - \lim x_k = 0$.

Definition 6 [17,18]. Let I be an admissible ideal of $\mathbb{N}$ and let $\theta = (k_r)$ be lacunary sequence. Then a sequence (x_k) is said to be *lacunary I-Cauchy* if there exists a subsequence $\left(x'_k(r)\right)$ of (x_k) such that $k'(r) \in J_r$ for each $r, \lim_{r\to\infty} x_{k'(r)} = L$ and for every $\varepsilon > 0$ such that

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in J_r}|x_k - x_{k'(r)}| \geqslant \varepsilon\right\} \in I.$$

Definition 7 [17,18]. A lacunary sequence $\theta' = (k'(r))$ is said to be a lacunary refinement of the lacunary sequence $\theta = (k_r)$ if $(k_r) \subset (k'(r))$.

Throughout the paper, we shall denote by I is an admissible ideal of subsets of $\mathbb{N} \times \mathbb{N}$ and $\theta_{r,s} = (k_{r,s})$ a double lacunary sequence of positive real numbers, respectively, unless otherwise stated.

3. Lacunary convergence of double sequences

By the convergence of a double sequence we mean the convergence in the Pringsheim's sense [26]. A double sequence $x = (x_{k,l})$ has a *Pringsheim* limit L (denoted by $P - \lim x = L$) provided that given an $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$. We shall describe such an $x = (x_{k,l})$ more briefly as "*P* − *convergent*".

A double sequence $\bar{\theta} = \theta_{r,s} = \{(k_r, l_s)\}$ is called *double lacunary sequence* if there exist two increasing sequence of integers (k_r) and (l_s) such that

$k_o = 0, h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$

and

$l_o = 0, \overline{h_s} = l_s - l_{s-1} \to \infty$ as $s \to \infty$.

Let us denote $k_{r,s} = k_r l_s, h_{r,s} = h_r\overline{h_s}$ and $\theta_{r,s}$ is determined by

$J_{r,s} = \{(k,l) : k_{r-1} < k \leqslant k_r \text{ and } l_{s-1} < l \leqslant l_s\}$,

$$q_r = \frac{k_r}{k_{r-1}}, \overline{q_s} = \frac{l_s}{l_{s-1}} \text{ and } q_{r,s} = q_r\overline{q_s}$$

(for details on double lacunary sequences we refer to [27]).

Definition 8. A double sequence $x = (x_{k,l})$ is said to be $\theta_{r,s}$-convergent to $L \in \mathbb{R}$ if for every $\varepsilon > 0$ and there exist integers $n_0 \in \mathbb{N}$ such that

$$\frac{1}{h_{r,s}}\sum_{(k,l)\in J_{r,s}}|x_{k,l} - L| < \varepsilon$$

for all $r, s \geqslant n_0$. In this case, we write $\theta_{r,s} - \lim x = L$.

Theorem 1. *Let $x = (x_{k,l})$ be a double sequence. If $x = (x_{k,l})$ is $\theta_{r,s}$-convergent then $\theta_{r,s} - \lim x$ is unique.*

Proof. The proof of the theorem is starightforward, thus omitted. □

4. Lacunary ideal convergence of double sequences

Definition 9. Let $\theta_{r,s} = (k_{r,s})$ be a double lacunary sequence. Then a double sequence $(x_{k,l})$ is said to be $I_{\theta_{r,s}}$*-convergent* if for every $\varepsilon > 0$ such that

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}}\sum_{(k,l)\in J_{r,s}}|x_{k,l} - L| \geqslant \varepsilon\right\} \in I.$$

We write $I_{\theta_{r,s}} - \lim x_{k,l} = L$.

Definition 10. A double sequence $(x_{k,l})$ is said to be $I_{\theta_{r,s}}$*-null* if for every $\varepsilon > 0$ such that

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}}\sum_{(k,l)\in J_{r,s}}|x_{k,l}| \geqslant \varepsilon\right\} \in I.$$