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ORIGINAL ARTICLE

A note on fixed point theorems for fuzzy mappings

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Abstract In this paper, a common fixed point theorem for contractive type fuzzy mappings in a complete metric space is proved due to Cho (2005) [1]. Further an example is given for the results of Cho (2005) [1, Theorem 3.1] and Park and Jeong (1997) [2, Theorem 3.2] which are not satisfying the condition "for all $x, y \in X$" and have a fixed point.

AMS MATHEMATICS SUBJECT CLASSIFICATION: 47H10; 54H25

1. Introduction and preliminaries

Let (X, d) be a metric space. Fixed points for multivalued mapping $T : X \to 2^X$ are defined as $x \in Tx$ for some $x \in X$. Let $\mathcal{CB}(X)$ denote the set of all nonempty closed and bounded subsets of X. A multivalued mapping $T : X \to \mathcal{CB}(X)$ is called a contraction mapping if there exists $q \in (0, 1)$ such that

$$H(T(x), T(y)) \leqslant qd(x, y) \quad \text{forall } x, y \in X,$$

where the Hausdroff metric $H(A, B)$ on $\mathcal{CB}(X)$ is given by

$$H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\},$$
$$\text{where } d(x, C) = \inf_{y\in C} d(x, y)$$

for any nonempty closed and bounded subsets A, B and C of X and for any point $x \in X$.

A fuzzy set in X is a function with domain X and values in $[0, 1]$. If A is a fuzzy set and $x \in X$, then the function values $A(x)$ are called the grade of membership of x in A. Let $\mathcal{F}(X)$ be the collection of all fuzzy sets on X and let ${}^{\alpha}A = \{x \in X : A(x) \geqslant \alpha\}$ denote the α-cut of $A \in \mathcal{F}(X)$.The zero-cut of A is defined as the closure of the set $\{x \in X : A(x) > 0\}$.

A mapping F from X to $\mathcal{F}(Y)$ is called a fuzzy mapping if for each $x \in X$, $F(x)$ is a fuzzy set on Y and $F(x)(y)$ denotes the degree of membership of y in $F(x)$. Let X be a metric linear space and let $\mathcal{W}(X)$ denote the set of all fuzzy sets on X such that each of its α-cut is a nonempty compact and convex subset (approximate quantity)of X. A fuzzy mapping F from X to $\mathcal{W}(X)$ is called a fuzzy contraction mapping if there exists $q \in (0, 1)$ such that

$$D(F(x), F(y)) \leqslant qd(x, y) \text{ for each } x, y \in X,$$
$$\text{where } D(A, B) = \sup_{\alpha} H({}^{\alpha}A, {}^{\alpha}B)$$

Define $p_{\alpha}(A, B) = \inf_{x\in {}^{\alpha}A, y\in {}^{\alpha}B} d(x, y)$ and $p(A, B) = \sup_{\alpha} p_{\alpha}(A, B)$ for any fuzzy sets $A, B \in \mathcal{W}(X)$.

It is known that p_{α} is non-decreasing function of α.

Heilpern [3] first introduced the concept of fuzzy mappings and proved a fixed point theorem for fuzzy contraction

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mappings which is a fuzzy analogue of the fixed point theorem of Nadler [4]. Bose and Sahani [5] extended the Heilpern's result for a pair of generalized fuzzy contraction mappings. Marudai and Srinivasan [6] generalized the Heilpern's result using the Nadler's result. They also obtained a nontrivial generalization of the Nadler's fixed point theorem for fuzzy contraction mappings under weaker settings. Further Vijayaraju and Marudai [7] generalize the result of Bose and Mukerjee [8] for contractive type fuzzy mappings in complete metric spaces. The significance of these results is assuming "each of its α-cut of fuzzy set is nonempty closed and bounded subset of X" instead of approximate quantity of X". Akbar Azam and Muhammad Arshad [9] proved the result of Vijayaraju and Marudai [7, Theorem 3.1] is incomplete and corrected the proof in right direction. In this paper a common fixed point theorem for contractive type fuzzy mappings in complete metric space due to Cho [1] is proved by using the concept of Vijayaraju and Marudai [7]. Further an example is given for the results of Cho [1, Theorem 3.1] and Park and Jeong [2, Theorem 3.2] which are not satisfying the condition "for all $x, y \in X$" and have a fixed point.

2. Main results

The following lemma due to Nadler [4] is the main key of our result.

Lemma 2.1 [4]. *Let (X, d) be a metric space and $A, B \in \mathcal{CB}(X)$, then for each $a \in A, k > 0$ there exists an element $b \in B$ such that $d(a, b) \leqslant H(A, B) + k$.*

Cho [1] and Park and Jeong [2] proved some fixed point theorems for fuzzy mappings from X to $W(X)$ under the contractive type conditions in complete metric space. The following example shows that the condition "for all $x, y \in X$" fails for the results [1, Theorem 3.1] and [2, Theorem 3.2].

Theorem 2.2 [1]. *Let $F, G : X \to W(X)$ be fuzzy mappings satisfying the following condition: There exists $k \in (0, 1)$ such that*

$$D(Fx, Gy) \leqslant \frac{k}{\sqrt{2}}\{p(x, Fx)p(y, Gy) + p(y, Gy)d(x, y)\}^{\frac{1}{2}} \qquad (*)$$

for all $x, y \in X$. Then F and G have a common fixed point.

Theorem 2.3 [2]. *Let $F, G : X \to W(X)$ be fuzzy mappings satisfying the following condition: There exists $k \in (0, 1)$ such that*

$$D(Fx, Gy) \leqslant k\{p(x, Fx)p(y, Gy)\}^{\frac{1}{2}} \qquad (**)$$

for all $x, y \in X$. Then F and G have a common fixed point.

Example 2.4. Let$X = [0, 1]$.For$x, y \in X, d(x, y) = |x - y|, \alpha \in (0, 1]$. Define $F, G : X \to W(X)$ by

$$F(0)(z) = \begin{cases} 1, & z = 0 \\ \frac{1}{2}, & 0 < z \leqslant 1/50 \\ 0, & z > 1/50 \end{cases} \quad G(0)(z) = \begin{cases} 1, & z = 0 \\ 1/4, & 0 < z \leqslant 1/100 \\ 0, & z > 1/100 \end{cases}$$

$$F(x)(z) = \begin{cases} \alpha, & 0 \leqslant z \leqslant x/25 \\ \frac{\alpha}{2}, & x/25 < z \leqslant x/10 \\ 0, & z > x/10 \end{cases} \quad G(x)(z) = \begin{cases} \alpha, & 0 \leqslant z \leqslant x/20 \\ \frac{\alpha}{2}, & x/20 < z \leqslant x/10 \\ 0, & z > x/10 \end{cases}$$

Here ${}^{1}F(x) = {}^{1}G(x) = \{0\}$ and ${}^{\alpha}F(x) = [0, x/25]$ and ${}^{\alpha}G(x) = [0, x/20]$

$$D(F(x), G(y)) = sup_{\alpha}H({}^{\alpha}F(x), {}^{\alpha}G(y)) = |x/20 - y/25|$$
$$\leqslant \frac{k}{\sqrt{2}}[|x - x/25|.|y - y/20| + |y - y/20||x - y|]^{\frac{1}{2}}$$

For $x = y, F$ and G satisfy all the conditions of Theorem 2.2 and 0 is the common fixed point of F and G.

For $x \neq y$, the condition $(*)$ fails for taking the values $x = 1, y = 0$.

Similarly the condition $(**)$ of Theorem 2.3 fails also.

From the above example, we observe that Theorem 2.2 holds for assuming the condition for all $x \in X$ and for all nonzero values of y in X and Theorem 2.3 holds for assuming the condition for all nonzero values $x, y \in X$.

Next a common fixed theorem for fuzzy mappings is proved due to Cho [1].

Theorem 2.5 [1]. *Let $F, G : X \to W(X)$ be fuzzy mappings satisfying the following condition: There exist $\alpha, \beta > 0$ such that $\alpha + \beta < 1$ and*

$$D(Fx, Gy) \leqslant \frac{\alpha p(y, Gy))[(1 + p(x, Fx))p(x, Fx)]^{\frac{1}{2}}}{1 + 2d(x, y)} + \beta d(x, y),$$

for all $x, y \in X$. Then F and G have a common fixed point.

Theorem 2.6. *Let (X, d) be a complete metric space and let F_1 and F_2 be fuzzy mappings from X to $\mathcal{F}(X)$ satisfying the following condition:*

(i) *For each $x, y \in X$, there exists $\alpha(x), \alpha(y) \in (0, 1]$ such that ${}^{\alpha(x)}F_1(x)$ and ${}^{\alpha(y)}F_2(y)$ are nonempty closed bounded subsets of X.*

(ii)

$$H({}^{\alpha(x)}F_1(x), {}^{\alpha(y)}F_2(y)) \leqslant \frac{a_1 d(y, {}^{\alpha(y)}F_2(y))[\{1 + d(x, {}^{\alpha(x)}F_1(x))\}d(x, {}^{\alpha(x)}F_1(x))]^{\frac{1}{2}}}{1 + 2d(x, y)} + a_2 d(x, y),$$

where $a_1, a_2 > 0$ and $a_1 + a_2 < 1$.

Then there exists $z \in X$ such that $z \in {}^{\alpha(z)}F_1(z) \cap {}^{\alpha(z)}F_2(z)$.

Proof. Let $x_0 \in X$. Then by condition (i), there exists $\alpha_1 \in (0, 1]$ such that ${}^{\alpha_1}F_1(x_0)$ is a nonempty closed bounded subset of X.

Choose $x_1 \in {}^{\alpha_1}F_1(x_0)$.

For this x_1, there exists $\alpha_2 \in (0, 1]$ such that ${}^{\alpha_2}F_2(x_1)$ is a nonempty closed bounded subset of X. Since ${}^{\alpha_1}F_1(x_0)$ and ${}^{\alpha_2}F_2(x_1)$ are nonempty closed bounded subsets of X and by Lemma 2.1, there exists $x_2 \in {}^{\alpha_2}F_2(x_1)$ such that