Journal of the Egyptian Mathematical Society (2016) 24, 77–82

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Fixed point theorems under Pata-type conditions in metric spaces

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Received 11 July 2014; accepted 2 September 2014
Available online 11 February 2015

KEYWORDS

Fixed point;
Chatterjea-type maps;
Common fixed point;
Coupled fixed point;
Pata-type condition

Abstract In this paper, we prove a generalization of Chatterjea's fixed point theorem, based on a recent result of Pata. Also, we establish common fixed point results of Pata-type for two maps, as well as a coupled fixed point result in ordered metric spaces. An example is given to show that new results are different from the known ones.

2010 MATHEMATICS SUBJECT CLASSIFICATION: Primary 47H10; Secondary 47H09

1. Introduction and preliminaries

Throughout this paper, (X, d) will be a given complete metric space. Let us select an arbitrary point $x_0 \in X$, and call it the "zero of X"; further, denote

$$\|x\| = d(x, x_0), \quad \text{for all } x \in X.$$

It will be clear that the obtained results do not depend on the particular choice of point x_0. Also, $\psi : [0, 1] \to [0, \infty)$ will be a fixed increasing function, continuous at zero, with $\psi(0) = 0$.

In a recent paper [1], Pata obtained the following refinement of the classical Banach Contraction Principle.

Theorem 1.1 [1]. *Let $f : X \to X$ and let $\Lambda \geqslant 0, \alpha \geqslant 1$ and $\beta \in [0, \alpha]$ be fixed constants. If the inequality*

$$d(fx, fy) \leqslant (1 - \varepsilon)d(x, y) + \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + \|x\| + \|y\|]^\beta \qquad (1.1)$$

is satisfied for every $\varepsilon \in [0, 1]$ and all $x, y \in X$, then f has a unique fixed point $z \in X$. Furthermore, the sequence $\{f^n x_0\}$ converges to z.

Chakraborty and Samanta extended in [2] the result of Pata to the case of Kannan-type contractive condition.

In this paper, we prove a further extension of Pata's result, using contractive condition of Chatterjea's type [3,4]. Also, we establish common fixed point results of Pata-type for two maps, as well as a coupled fixed point result in ordered metric spaces. An example is given to show that new results are different from the known ones.

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Peer review under responsibility of Egyptian Mathematical Society.

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http://dx.doi.org/10.1016/j.joems.2014.09.001
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1.1. An auxiliary result

Assertions similar to the following lemma were used (and proved) in the course of proofs of several fixed point results in various papers.

Lemma 1.1 [5]. *Let (X, d) be a metric space and let $\{y_n\}$ be a sequence in X such that $d(y_{n+1}, y_n)$ is nonincreasing and that*

$$\lim_{n\to\infty} d(y_{n+1}, y_n) = 0.$$

If $\{y_{2n}\}$ is not a Cauchy sequence then there exist a $\delta > 0$ and two strictly increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following sequences tend to δ when $k \to \infty$:

$$\begin{aligned} &d(y_{2m_k}, y_{2n_k}), \quad d(y_{2m_k}, y_{2n_k+1}), \quad d(y_{2m_k-1}, y_{2n_k}), \\ &d(y_{2m_k-1}, y_{2n_k+1}), \quad d(y_{2m_k+1}, y_{2n_k+1}). \end{aligned} \tag{1.2}$$

2. A Chatterjea-type fixed point result

Theorem 2.1. *Let $f : X \to X$ and let $\Lambda \geqslant 0, \alpha \geqslant 1$ and $\beta \in [0, \alpha]$ be fixed constants. If the inequality*

$$\begin{aligned} d(fx, fy) \leqslant &\frac{1-\varepsilon}{2}(d(x, fy) + d(y, fx)) \\ &+ \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + \|x\| + \|y\| + \|fx\| + \|fy\|]^\beta \end{aligned} \tag{2.1}$$

is satisfied for every $\varepsilon \in [0, 1]$ and all $x, y \in X$, then f has a unique fixed point $z \in X$.

Proof.

1. Uniqueness.For any two fixed $u, v \in X$, we can write (2.1) in the form

   $$d(fu, fv) \leqslant \frac{1-\varepsilon}{2}(d(u, fv) + d(v, fu)) + K\varepsilon\psi(\varepsilon), \qquad K > 0.$$

   If $fu = u$ and $fv = v$ then

   $$d(u, v) \leqslant K\psi(\varepsilon),$$

   for all $\varepsilon \in (0, 1]$, which implies that $d(u, v) = 0$.
2. Existence of z.
   Starting from x_0, we introduce the sequences

   $$x_n = fx_{n-1} = f^n x_0 \text{ and } c_n = \|x_n\|.$$

2.1. First, we have that the sequence $d(x_{n+1}, x_n)$ is nonincreasing, that is

$$d(x_{n+1}, x_n) \leqslant d(x_n, x_{n-1}) \leqslant \cdots \leqslant d(x_1, x_0), \tag{2.2}$$

for all $n \in \mathbb{N}$.
Indeed, putting $\varepsilon = 0, x = x_n, y = x_{n-1}$ in (2.1), we obtain (2.2).

2.2. The sequence $\{c_n\}$ is bounded.
Using (2.2), we deduce the following estimate

$$\begin{aligned} c_n = d(x_n, x_0) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_1) + d(x_1, x_0) \\ &\leqslant d(x_{n+1}, x_1) + 2c_1 = d(fx_n, fx_0) + 2c_1. \end{aligned}$$

Therefore, we infer from (2.1) that

$$\begin{aligned} c_n \leqslant &\frac{1-\varepsilon}{2}[d(x_n, x_1) + d(x_{n+1}, x_0)] \\ &+ \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + \|x_n\| + \|x_{n+1}\| + \|x_1\|]^\beta + 2c_1. \end{aligned}$$

Using $d(x_n, x_1) \leqslant d(x_n, x_0) + d(x_0, x_1), d(x_{n+1}, x_0) \leqslant d(x_{n+1}, x_n) + d(x_n, x_0)$ and (2.2), as $\beta \leqslant \alpha$, the previous inequality implies that

$$c_n \leqslant (1-\varepsilon)(c_n + c_1) + \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + 2c_n + 2c_1]^\alpha + 2c_1$$

Now,

$$[1 + 2c_n + 2c_1]^\alpha \leqslant (1 + 2c_n)^\alpha(1 + 2c_1)^\alpha \leq 2^\alpha c_n^\alpha(1 + 2c_1)^\alpha,$$

which implies that

$$c_n \leqslant (1-\varepsilon)c_n + a\varepsilon^\alpha\psi(\varepsilon)c_n^\alpha + b,$$

for some $a, b > 0$. Hence,

$$\varepsilon c_n \leqslant a\varepsilon^\alpha\psi(\varepsilon)c_n^\alpha + b.$$

Now, for the same reason as in [1], it follows that the sequence $\{c_n\}$ is bounded.

2.3. $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.
For all $\varepsilon \in (0, 1]$ and for $x = x_n$, $y = x_{n-1}$ we have

$$\begin{aligned} d(x_{n+1}, x_n) = d(fx_n, fx_{n-1}) \leqslant &\frac{1-\varepsilon}{2}(d(x_n, x_n) + d(x_{n-1}, x_{n+1})) \\ &+ \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + 2\|x_n\| + \|x_{n-1}\| \\ &+ \|x_{n+1}\|]^\beta \leqslant \frac{1-\varepsilon}{2}(d(x_{n-1}, x_n) \\ &+ d(x_n, x_{n+1})) + K\varepsilon\psi(\varepsilon), \quad K > 0. \end{aligned} \tag{2.3}$$

If $\lim_{n\to\infty} d(x_{n+1}, x_n) = d^* > 0$, it follows from (2.3) that

$$d^* \leqslant K\psi(\varepsilon),$$

that is $d^* = 0$. A contradiction.

2.4. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.
If it is not the case, choose $\delta > 0$, $\{m_k\}$ and $\{n_k\}$ as in Lemma 2.1. Putting $x = x_{2m(k)-1}$, $y = x_{2n(k)}$ in (2.1), we obtain

$$\begin{aligned} d(x_{2m(k)}, x_{2n(k)+1}) \leqslant &\frac{1-\varepsilon}{2}(d(x_{2m(k)-1}, x_{2n(k)+1}) \\ &+ d(x_{2m(k)}, x_{2n(k)})) + K\varepsilon\psi(\varepsilon), \end{aligned} \tag{2.4}$$

where $d(x_{2m(k)}, x_{2n(k)+1}) \to \delta$, $d(x_{2m(k)-1}, x_{2n(k)+1}) \to \delta$ and $d(x_{2m(k)}, x_{2n(k)}) \to \delta$. Letting $k \to \infty$ in (2.4), we obtain

$$\delta \leqslant K\psi(\varepsilon),$$

that is $\delta = 0$, a contradiction.
Taking into account the completeness of (X, d), we can now guarantee the existence of some $z \in X$ to which $\{x_n\}$ converges.Finally, all that remains to show is:

2.5. z is a fixed point for f.
For this we observe that, for all $n \in \mathbb{N}$ and for $\varepsilon = 0$,

$$\begin{aligned} d(fz, z) &\leqslant d(fz, x_{n+1}) + d(x_{n+1}, z) = d(fz, fx_n) + d(x_{n+1}, z) \\ &\leqslant \frac{1}{2}(d(z, x_{n+1}) + d(fz, x_n)) + d(x_{n+1}, z). \end{aligned}$$

Hence, $d(fz, z) \leqslant \frac{1}{2} d(fz, z)$, that is $fz = z$, which is the required result. □

The classical Chatterjea's result [3] is a consequence of Theorem 2.1, since the condition

$$d(fx, fy) \leqslant \frac{\lambda}{2}(d(x, fy) + d(y, fx))$$

for some $\lambda \in [0, 1)$ and all $x, y \in X$, implies condition (2.1). This can be proved in the same way as in [1, Section 3], or [2, Section 3].