Journal of the Egyptian Mathematical Society (2016) **24**, 86–91

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Numerical solution of time-dependent diffusion equations with nonlocal boundary conditions via a fast matrix approach

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Received 21 December 2013; revised 1 June 2014; accepted 18 June 2014
Available online 18 July 2014

KEYWORDS

One-dimensional parabolic equation;
Nonlocal boundary conditions;
Taylor approximation;
Operational matrices;
Krylov subspace iterative methods;
Restarted GMRES

Abstract This article contributes a matrix approach by using Taylor approximation to obtain the numerical solution of one-dimensional time-dependent parabolic partial differential equations (PDEs) subject to nonlocal boundary integral conditions. We first impose the initial and boundary conditions to the main problems and then reach to the associated integro-PDEs. By using operational matrices and also the completeness of the monomials basis, the obtained integro-PDEs will be reduced to the generalized Sylvester equations. For solving these algebraic systems, we apply a famous technique in Krylov subspace iterative methods. A numerical example is considered to show the efficiency of the proposed idea.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 65M70; 35R10; 41A58

1. Introduction

One dimensional parabolic partial differential equations (PDEs) have an extensive application in the study of problems in engineering and applied sciences. It should be mentioned that, such PDEs together with classical boundary conditions have received considerable attention in research works. However, these PDEs with nonlocal boundary conditions were studied by researchers in the literature, but extensions and modifications of the existing methods should be explored to obtain more accurate solutions. The usual numerical methods for PDEs subject to the nonlocal boundary conditions are finite difference methods [1–3], Galerkin techniques [4], collocation approaches [5], Tau schemes [6] and reproducing kernel space methods [7]. Moreover, some other new methods were considered in [8–11].

It should be noted that, in all of the research works that are based on the operational matrices, the basic PDEs (with classical boundary conditions) were finally transformed into the matrix–vector algebraic system $Ax = b$, which can be solved

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Peer review under responsibility of Egyptian Mathematical Society.

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http://dx.doi.org/10.1016/j.joems.2014.06.018

by robust iterative solvers such as Krylov subspace iterative methods (e.g., restarted GMRES and Bi-CGSTAB methods). For this purpose, one can use simple MATLAB commands for applying these iterative solvers. On the other hand, if the PDEs contain nonlocal boundary conditions, they may be transformed into the associated generalized Sylvester equations by using operational matrices. Since for solving such generalized Sylvester equations, there is no MATLAB commands, we should extend Krylov subspace iterative methods. Moreover, Taylor matrix approaches have had no results for solving PDEs subject to non-classical boundary conditions. These are basic motivations of the paper. In this paper, we present a new matrix method for solving one-dimensional parabolic time-dependent diffusion equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+K(x,t),\qquad 0<x<1,\quad 0<t\leqslant 1, \tag{1}$$

with the initial condition

$$u(x,0)=f(x),\qquad 0<x<1, \tag{2}$$

and the nonlocal boundary conditions

$$\begin{aligned} u(0,t)&=\int_0^1 \rho(x)u(x,t)dx, \qquad 0<t\leqslant 1,\\ u(1,t)&=\int_0^1 \psi(x)u(x,t)dx, \qquad 0<t\leqslant 1, \end{aligned} \tag{3}$$

where K, f, ρ and ψ are known functions, while the function u should be determined. It should be mentioned that we develop a new matrix approach, which was previously examined in [12–15], for solving one-dimensional parabolic PDEs with nonlocal boundary conditions. Some straightforward manipulations, enable us to impose the initial and boundary conditions (2) and (3) to the main problem. Thus, completeness of monomials basis together with the operational matrices of differentiation and integration can be used to reduce the main problem to the associated generalized Sylvester equations. Actually this is the first operational matrix approach for which the final associated algebraic system (i.e., generalized Sylvester equations) will be considered with more details.

2. Method of the solution

In this section, the basic Eq. (1) subject to the initial and boundary conditions (2) and (3) will be transformed into the associated integro-PDE by some straightforward manipulations. Then, completeness of monomials basis together with the operational matrices of differentiation and integration can be used to reduce the main problem to the associated generalized Sylvester equations. For this purpose, we should recall the operational matrices as follows

$$X'(x)=\begin{bmatrix}1\\x\\\vdots\\x^{N-1}\\x^N\end{bmatrix}'=\overbrace{\begin{bmatrix}0&0&0&\cdots&0\\1&0&0&\cdots&0\\0&2&0&\cdots&0\\\vdots&\vdots&\ddots&\ddots&\vdots\\0&0&\cdots&N&0\end{bmatrix}}^{M}\begin{bmatrix}1\\x\\\vdots\\x^{N-1}\\x^N\end{bmatrix}, \tag{4}$$

$$\int_0^x X(x')dx'=\int_0^x\begin{bmatrix}1\\x'\\\vdots\\x'^{N-1}\\x'^N\end{bmatrix}dx'\approx\overbrace{\begin{bmatrix}0&1&0&\cdots&0\\0&0&\frac{1}{2}&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&\frac{1}{N}\\0&0&0&\cdots&0\end{bmatrix}}^{P}\begin{bmatrix}1\\x\\\vdots\\x^{N-1}\\x^N\end{bmatrix}, \tag{5}$$

where M and P are operational matrices of differentiation and integration, respectively. It should be recalled that $\int_0^1 X(x)X^T(x)dx=Q$, where $Q=hilb(N+1)$ is the Hilbert matrix of dimension $N+1$. Throughout of the paper, Q denotes the hilbert matrix of dimension $N+1$ and we do not show its index for clarity of presentation. Now, one can rewrite the basic Eq. (1) in the form

$$u_{xx}(x,t)=u_t(x,t)-K(x,t).$$

So direct integration from both sides of the above equation with respect to x in the interval $[0,x]$ yields

$$\begin{aligned} u_x(x,t)&=u_x(0,t)+\int_0^x u_{xx}(x',t)dx'\\ &=u_x(0,t)+\int_0^x (u_t(x',t)-K(x',t))dx'. \end{aligned} \tag{6}$$

On the other hand, by assuming $u(0,t)=A(t)$, one can write

$$u(x,t)=A(t)+\int_0^x u_x(x',t)dx'. \tag{7}$$

From (6) and (7) one can conclude that

$$u(x,t)=A(t)+xu_x(0,t)+\int_0^x\int_0^{x'}(u_t(x'',t)-K(x'',t))dx''dx'. \tag{8}$$

We suppose that $u(1,t)=B(t)$, and hence

$$\begin{aligned} B(t)&=u(1,t)\\ &=A(t)+u_x(0,t)+\int_0^1\int_0^x(u_t(x',t)-K(x',t))dx'dx. \end{aligned}$$

The above equation can be rewritten in the form

$$\begin{aligned} u_x(0,t)&=B(t)\\ &\quad-\left(A(t)+\int_0^1\int_0^x(u_t(x',t)-K(x',t))dx'dx\right). \end{aligned} \tag{9}$$

Replacing (9) into (8) yields

$$\begin{aligned} u(x,t)&=(1-x)A(t)+xB(t)-x\int_0^1\int_0^x(u_t(x',t)\\ &\quad-K(x',t))dx'dx+\int_0^x\int_0^{x'}(u_t(x'',t)\\ &\quad-K(x'',t))dx''dx'. \end{aligned} \tag{10}$$

For imposing the initial condition (2), we should differentiate both sides of (10) with respect to t in the following form

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t}&=\frac{\partial}{\partial t}[(1-x)A(t)+xB(t)\\ &\quad-x\int_0^1\int_0^x(u_t(x',t)-K(x',t))dx'dx\\ &\quad+\int_0^x\int_0^{x'}(u_t(x'',t)-K(x'',t))dx''dx' \end{aligned}$$