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ORIGINAL ARTICLE

Direct and inverse theorems for Bernstein polynomials with inner singularities

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Abstract We introduce a new type of Bernstein polynomials, which can be used to approximate the functions with inner singularities. The direct and inverse results of the weighted approximation of this new type of combinations are obtained.

1. Introduction

The set of all continuous functions, defined on the interval I, is denoted by $C(I)$. For any $f \in C([0,1])$, the corresponding Bernstein polynomials are defined as follows:

$$B_n(f,x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x),$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \ k = 0,1,2,\ldots,n, \ x \in [0,1].$$

Let $\bar{w}(x) = |x-\xi|^{\alpha}, \ 0 < \xi < 1, \ \alpha > 0$ and $C_{\bar{w}} := \{f \in C([0,1] \setminus \{\xi\}) : \lim_{x\to\xi} (\bar{w}f)(x) = 0\}$. The norm in $C_{\bar{w}}$ is defined by $\|f\|_{C_{\bar{w}}} := \|\bar{w}f\| = \sup_{0 \leqslant x \leqslant 1} |(\bar{w}f)(x)|$. Define

$$W_{\phi}^2 := \{f \in C_{\bar{w}} : f' \in A.C.((0,1)), \ \|\bar{w}\phi^2 f''\| < \infty\},$$
$$W_{\bar{w},\lambda}^2 := \{f \in C_{\bar{w}} : f' \in A.C.((0,1)), \ \|\bar{w}\varphi^{2\lambda} f''\| < \infty\}.$$

For $f \in C_{\bar{w}}$, the weighted modulus of smoothness is defined by

$$\omega_{\phi}^2(f,t)_{\bar{w}} := \sup_{0<h \leqslant t} \sup_{0 \leqslant x \leqslant 1} |\bar{w}(x) \Delta_{h\phi(x)}^2 f(x)|,$$

where

$$\Delta_{h\phi}^2 f(x) = f(x + h\phi(x)) - 2f(x) + f(x - h\phi(x)),$$

and $\varphi(x) = \sqrt{x(1-x)}, \ \delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n}}$.

Recently Felten showed the following two theorems in [1]:

Theorem A. *Let $\varphi(x) = \sqrt{x(1-x)}$ and let $\phi : [0,1] \to R$, $\phi \neq 0$ be an admissible step-weight function of the Ditzian–Totik modulus of smoothness [4] such that ϕ^2 and φ^2/ϕ^2 are concave. Then, for $f \in C[0,1]$ and $0 < \alpha < 2$,*

$$|B_n(f,x) - f(x)| \leqslant \omega_{\phi}^2\left(f, n^{-1/2} \frac{\varphi(x)}{\phi(x)}\right).$$

Theorem B. *Let $\varphi(x) = \sqrt{x(1-x)}$ and let $\phi : [0,1] \to R$, $\phi \neq 0$ be an admissible step-weight function of the Ditzian–Totik*

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modulus of smoothness such that ϕ^2 *and* φ^2/ϕ^2 *are concave. Then, for* $f \in C[0,1]$ *and* $0 < \alpha < 2$,

$$|B_n(f,x) - f(x)| = O\left(\left(n^{-1/2}\frac{\varphi(x)}{\phi(x)}\right)^{\alpha}\right)$$

implies $\omega_{\phi}^2(f,t) = O(t^{\alpha})$.

Approximation properties of Bernstein polynomials have been studied very well [2–5]. In order to approximate the functions with singularities, Della Vecchia et al. [3] introduced some kinds of modified Bernstein polynomials. Throughout the paper, C denotes a positive constant independent of n and x, which may be different in different cases.

Let ϕ: $[0,1] \to R$, $\phi \neq 0$ be an admissible step-weight function of the Ditzian–Totik modulus of smoothness, that is, ϕ satisfies the following conditions:

- (**I**) For every proper subinterval $[a,b] \subseteq [0,1]$ there exists a constant $C_1 \equiv C(a,b) > 0$ such that $C_1^{-1} \leqslant \phi(x) \leqslant C_1$ for $x \in [a,b]$.
- (**II**) There are two numbers $\beta(0) \geqslant 0$ and $\beta(1) \geqslant 0$ for which

$$\phi(x) \sim \begin{cases} x^{\beta(0)}, & \text{as } x \to 0+, \\ (1-x)^{\beta(1)}, & \text{as } x \to 1-. \end{cases}$$

($X \sim Y$ means $C^{-1}Y \leqslant X \leqslant CY$for some C).

Combining conditions (**I**) and (**II**) on ϕ, we can deduce that

$$C^{-1}\phi_2(x) \leqslant \phi(x) \leqslant C\phi_2(x), \ x \in [0,1],$$

where $\phi_2(x) = x^{\beta(0)}(1-x)^{\beta(1)}$.

2. The main results

Let

$$\psi(x) = \begin{cases} 10x^3 - 15x^4 + 6x^5, & 0 < x < 1, \\ 0, & x \leqslant 0, \\ 1, & x \geqslant 1. \end{cases}$$

Obviously, ψ is non-decreasing on the real axis, $\psi \in C^2((-\infty,+\infty))$, $\psi^{(i)}(0) = 0$, $i = 0, 1, 2$. $\psi^{(i)}(1) = 0$, $i = 1{,}2$ and $\psi(1) = 1$. Further, let

$$x_1 = \frac{[n\xi - 2\sqrt{n}]}{n}, \ x_2 = \frac{[n\xi - \sqrt{n}]}{n}, \ x_3 = \frac{[n\xi + \sqrt{n}]}{n}, \ x_4 = \frac{[n\xi + 2\sqrt{n}]}{n},$$

and

$$\bar{\psi}_1(x) = \psi\left(\frac{x - x_1}{x_2 - x_1}\right), \ \bar{\psi}_2(x) = \psi\left(\frac{x - x_3}{x_4 - x_3}\right).$$

Consider

$$P(x) := \frac{x - x_4}{x_1 - x_4}f(x_1) + \frac{x_1 - x}{x_1 - x_4}f(x_4),$$

the linear function joining the points $(x_1,f(x_1))$ and $(x_4,f(x_4))$. And let

$$\overline{F}_n(f,x) := \overline{F}_n(x) = f(x)(1 - \bar{\psi}_1(x) + \bar{\psi}_2(x)) + \bar{\psi}_1(x)(1 - \bar{\psi}_2(x))P(x).$$

From the above definitions it follows that

$$\overline{F}_n(f,x) = \begin{cases} f(x), & x \in [0,x_1] \cup [x_4,1], \\ f(x)(1 - \bar{\psi}_1(x)) + \bar{\psi}_1(x)P(x), & x \in [x_1,x_2], \\ P(x), & x \in [x_2,x_3], \\ P(x)(1 - \bar{\psi}_2(x)) + \bar{\psi}_2(x)f(x), & x \in [x_3,x_4]. \end{cases}$$

Evidently, $\overline{F}_n$ is a positive linear polynomials which depends on the functions values $f(k/n)$, $0 \leqslant k/n \leqslant x_2$ or $x_3 \leqslant k/n \leqslant 1$, it reproduces linear functions, and $\overline{F}_n \in C^2([0,1])$ provided $f \in W_{\phi}^2$. Now for every $f \in C_{\bar{w}}$ define the Bernstein type polynomials

$$\begin{aligned} \overline{B}_n(f,x) &:= B_n(\overline{F}_n(f),x) \\ &= \sum_{k/n \in [0,x_1] \cup [x_4,1]} p_{n,k}(x) f\left(\frac{k}{n}\right) + \sum_{x_2 < k/n < x_3} p_{n,k}(x) P\left(\frac{k}{n}\right) \\ &\quad + \sum_{x_1 < k/n < x_2} p_{n,k}(x)\left\{f\left(\frac{k}{n}\right)\left(1 - \bar{\psi}_1\left(\frac{k}{n}\right)\right) + \bar{\psi}_1\left(\frac{k}{n}\right)P\left(\frac{k}{n}\right)\right\} \\ &\quad + \sum_{x_3 < k/n < x_4} p_{n,k}(x)\left\{P\left(\frac{k}{n}\right)\left(1 - \bar{\psi}_2\left(\frac{k}{n}\right)\right) + \bar{\psi}_2\left(\frac{k}{n}\right)f\left(\frac{k}{n}\right)\right\}. \end{aligned} \tag{2.1}$$

Obviously, $\overline{B}_n$ is a positive linear polynomials, $\overline{B}_n(f)$ is a polynomial of degree at most n, it preserves linear functions, and depends only on the function values $f(k/n)$, $k/n \in [0,x_2] \cup [x_3,1]$. Now we state our main results as follows:

Theorem 1. *If* $\alpha > 0$, *for any* $f \in C_{\bar{w}}$, *we have*

$$\left\|\bar{w}\overline{B}_n''(f)\right\| \leqslant Cn^2\|\bar{w}f\|. \tag{2.2}$$

Theorem 2. *For any* $\alpha > 0$, $\min\{\beta(0),\beta(1)\} \geqslant \frac{1}{2}$, $0 < \xi < 1$, *we have*

$$|\bar{w}(x)\phi^2(x)\overline{B}_n''(f,x)| \leqslant \begin{cases} Cn\|\bar{w}f\|, & f \in C_{\bar{w}}, \\ C\|\bar{w}\phi^2 f''\|, & f \in W_{\phi}^2. \end{cases} \tag{2.3}$$

Theorem 3. *For* $f \in C_{\bar{w}}$, $0 < \xi < 1$, $\alpha > 0$, $\min\{\beta(0),\beta(1)\} \geqslant \frac{1}{2}$, $\alpha_0 \in (0,2)$, *we have*

$$\bar{w}(x)|f(x) - \overline{B}_n(f,x)| = O\left(\left(n^{-\frac{1}{2}}\phi^{-1}(x)\delta_n(x)\right)^{\alpha_0}\right) \Longleftrightarrow \omega_{\phi}^2(f,t)_{\bar{w}} = O(t^{\alpha_0}).$$

3. Lemmas

Lemma 1. [7] *For any non-negative real* u *and* v, *we have*

$$\sum_{k=1}^{n-1}\left(\frac{k}{n}\right)^{-u}\left(1 - \frac{k}{n}\right)^{-v} p_{n,k}(x) \leqslant Cx^{-u}(1-x)^{-v}. \tag{3.1}$$

Lemma 2. [3] *For any* $\alpha \geq 0$, $f \in C_{\bar{w}}$, *we have*

$$\|\bar{w}\overline{B}_n(f)\| \leqslant C\|\bar{w}f\|. \tag{3.2}$$

Lemma 3. [6] *Let* $\min\{\beta(0),\beta(1)\} \geqslant \frac{1}{2}$, *then for* $0 < t < \frac{1}{4}$ *and* $t < x < 1 - t$, *we have*