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ORIGINAL ARTICLE

Application of non-polynomial spline to the solution of fifth-order boundary value problems in induction motor

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Abstract Non-polynomial spline functions of the form Span$\{1, x, x^2, x^3, x^4, x^5, cos(kx) + e^x\}$, where k can be real or pure imaginary, are used to find the numerical solution of linear fifth-order boundary value problems. The order of convergence of the method is observed to be of $O(h^2)$. A fifth order convergent method is defined with the help of improved end-conditions. Three examples are considered to show the reliability and efficiency of the method. The numerical results, obtained, endorse the improved order of convergence of the method.

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1. Introduction

The behavior of an induction motor with two rotor circuits is represented by a fifth-order differential equation model. This model contains two stator state variables, two rotor state variables and one shaft speed. Normally, two more variables must be added to account for the effects of a second rotor circuit representing deep bars, a starting cage or rotor distributed parameters. To avoid the computational burden of additional state variables when additional rotor circuits are required, model is often limited to the fifth order and rotor impedance is algebraically altered as function of rotor speed. This is done under the assumption that the frequency of rotor currents depends on rotor speed. This approach is efficient for the steady state response with sinusoidal voltage [1].

Fifth-order boundary value problems also arise in the mathematical modeling of viscoelastic fluids [2,3]. Agarwal presented the conditions for existence and uniqueness of solutions of such problems [4]. Caglar et al. [5] used sixth degree B-spline to solve fifth-order linear and non-linear boundary value problems. Siddiqi and Twizell [6–9] presented the solutions of 6th, 8th, 10th and 12th order boundary value problems using 6th, 8th, 10th and 12th degree splines, respectively. Siddiqi and Akram solved the fifth-order linear special case boundary value problem using sextic spline and non-polynomial spline technique [10,11] respectively. Noor and Mohyud-Din [12] used a decomposition method to find the solution of fifth-order boundary value problems in terms of convergent series. Viswanadham et al. [13] used collocation method with sixth degree B-splines as basis functions to solve fifth-order special case boundary value problems.

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Akram and Rehman [14] used reproducing kernel space method to find approximate solutions to fifth-order boundary value problems. Siddiqi et al. [15] used non-polynomial spline for the numerical solutions of fifth-order singularly perturbed boundary value problems. Viswanadham and Raju [16] developed a finite element method involving collocation method with quartic B-splines as basis functions.

In this paper, non-polynomial spline functions of the form

$$T_n = \text{Span}\{1, x, x^2, x^3, x^4, x^5, cos(kx) + e^x\}$$

are used to develop the technique for the solution of fifth-order boundary value problems. It is to be noted that k can be real or pure imaginary. The fifth-order boundary value problem (BVP) of the following form has been considered

$$\left.\begin{array}{l} y^{(5)}(x) + f(x)y(x) = g(x), \quad x \in [a,b] \\ y(a) = \alpha_0, \quad y(b) = \beta_0, \\ y'(a) = \alpha_1, \quad y'(b) = \beta_1, \\ y''(a) = \alpha_2. \end{array}\right\} \tag{1.1}$$

where α_0, α_1, α_2, β_0 and β_1 are finite real constants, also $f(x)$ and $g(x)$ are continuous on $[a,b]$.

2. Preliminary results

The interval $[a,b]$ of domain has been subdivided into n equal subintervals using the grid points $x_i = a + ih, i = 0, 1, \ldots, n$, where $h = \frac{(b-a)}{n}$.

Consider the following restriction S_i of S to each subinterval $[x_i, x_{i+1}], i = 0, 1, \ldots, n-1$,

$$\begin{aligned} S_i(x) = {} & a_i\{cosk(x - x_i) + e^{(x-x_i)}\} + b_i(x - x_i)^5 \\ & + c_i(x - x_i)^4 + d_i(x - x_i)^3 + e_i(x - x_i)^2 \\ & + f_i(x - x_i) + g_i \end{aligned} \tag{2.2}$$

Let

$$\left.\begin{array}{ll} y_i = S_i(x_i), & m_i = S_i^{(1)}(x_i), \\ M_i = S_i^{(3)}(x_i), & l_i = S_i^{(5)}(x_i), \end{array}\right\} \tag{2.3}$$

where $i = 0, 1, \ldots, n$.

Using (2.3), the coefficients in (2.2) are determined as

$$a_i = \frac{h^5(-l_i + l_{i+1})}{h^5(-1 + e^h) - \theta^5 + \theta^5 \cos\theta},$$

$$b_i = \frac{(e^h h^5 + \theta^5 \cos\theta)l_i - (h^5 + \theta^5)l_{i+1}}{120\{h^5(-1 + e^h) - \theta^5 + \theta^5 \cos\theta\}},$$

$$\begin{aligned} c_i = {} & \left[\frac{-h(-240 + 240e^h - 120h - 120e^h h + 20h^3 + 3e^h h^5)}{240\{h^5(-1 + e^h) - \theta^5 + \theta^5 \cos\theta\}} \right. \\ & \left. + \frac{h(120\theta + 20\theta^3 + 3\theta(40 - \theta^4)\cos\theta - 240\sin\theta)}{240\{h^5(-1 + e^h) - \theta^5 + \theta^5 \cos\theta\}}\right] l_i \\ & + \left[\frac{h(-240 + 240e^h - 120h - 120e^h h + 20h^3 + 3h^5)}{240\{h^5(-1 + e^h) - \theta^5 + \theta^5 \cos\theta\}} \right. \\ & \left. + \frac{h(-120\theta - 20\theta^3 + 3\theta^5 - 120\theta\cos\theta + 240\sin\theta)}{240\{h^5(-1 + e^h) - \theta^5 + \theta^5 \cos\theta\}}\right] l_{i+1} \\ & + \frac{m_i + m_{i+1}}{2h^3} - \frac{M_i}{12h} + \frac{y_i - y_{i+1}}{h^4}, \end{aligned}$$

$$d_i = \frac{1}{6}\left(\frac{h^2(h^3 - \theta^3)(l_i - l_{i+1})}{h^5(-1 + e^h) - \theta^5 + \theta^5 \cos\theta} + M_i\right),$$

$$\begin{aligned} e_i = {} & \left[\frac{h^3(-480 + 480e^h - 360h - 120e^h h - 20h^3 + e^h h^5)}{240\{h^5(-1 + e^h) - \theta^5 + \theta^5 \cos\theta\}} \right. \\ & \left. + \frac{h^3(-360\theta + 20\theta^3 + \theta(-120 + \theta^4)\cos\theta + 480\sin\theta)}{240\{h^5(-1 + e^h) - \theta^5 + \theta^5 \cos\theta\}}\right] l_i \\ & - \left[\frac{h^3(-480 + 480e^h - 360h - 120e^h h - 20h^3 + h^5)}{240\{h^5(-1 + e^h) - \theta^5 + \theta^5 \cos\theta\}} \right. \\ & \left. - \frac{h^3(-360\theta + 20\theta^3 + \theta^5 - 120\theta\cos\theta + 480\sin\theta)}{240\{h^5(-1 + e^h) - \theta^5 + \theta^5 \cos\theta\}}\right] l_{i+1} \\ & - \frac{3m_i + m_{i+1}}{2h} - \frac{M_i h}{12} - 2\frac{y_i - y_{i+1}}{h^2}, \end{aligned}$$

$$f_i = \frac{h^4(h + \theta)(l_i - l_{i+1})}{h^5(-1 + e^h) - \theta^5 + \theta^5 \cos\theta} + m_i,$$

$$g_i = \frac{h^5(l_i - l_{i+1})}{h^5(-1 + e^h) - \theta^5 + \theta^5 \cos\theta} + y_i, \tag{2.4}$$

where $\theta = kh$ and $i = 0, 1, \ldots, n-1$.

The following consistency relations are derived by applying the second, third and fourth derivative continuities at knots, i.e. $S_{i-1}^{(\lambda)}(x_i) = S_i^{(\lambda)}(x_i)$, where $\lambda = 2, 3$ and 4

$$\begin{aligned} & [h^5\{-960 + 960e^h - 360h - 600e^h h + 120e^h h^2 + 20h^3 - 3e^h h^5 \\ & -360\theta - 20\theta^3 - 3\theta(200 + \theta^4)\cos\theta - 120(-8 + \theta^2)\sin\theta\}l_{i-1} \\ & + h^5\{480 - 480e^h + 480e^h h - 120h^2 - 120e^h h^2 - 40h^3 + 3h^5 + e^h h^5 \\ & + 40\theta^3 + 3\theta^5 + \theta(480 + \theta^4)\cos\theta + 120(-4 + \theta^2)\sin\theta\}l_i \\ & + h^5\{480 - 480e^h + 360h + 120e^h h + 120h^2 + 20h^3 - h^5 + 360\theta \\ & - 20\theta^3 - \theta^5 + 120\theta\cos\theta - 480\sin\theta\}l_{i+1} + 20\{h^5(-1 + e^h) \\ & - \theta^5 + \theta^5\cos\theta\}\{h(-18m_{i-1} - 48m_i - 6m_{i+1}) \\ & + h^3(M_{i-1} - M_i) + 24(-2y_{i-1} + y_i + y_{i+1})\}] \\ & = 0, \quad i = 1, 2, \ldots, n-1, \end{aligned} \tag{2.5}$$

$$\begin{aligned} & \left[\frac{h^2\{120 - 120e^h + 60h + 60e^h h - 5h^3 - 5e^h h^3 + e^h h^5\}}{5\{h^5(-1 + e^h) - \theta^5 + \theta^5 \cos\theta\}} \right. \\ & \left. + \frac{h^2\{60\theta + 5\theta^3 + \theta(60 + 5\theta^2 + \theta^4)\cos\theta - 120\sin\theta\}}{5\{h^5(-1 + e^h) - \theta^5 + \theta^5 \cos\theta\}}\right] l_{i-1} \\ & - \left[\frac{h^2\{120 - 120e^h + 60h + 60e^h h - 5h^3 - 5e^h h^3 + h^5\}}{5\{h^5(-1 + e^h) - \theta^5 + \theta^5 \cos\theta\}} \right. \\ & \left. + \frac{h^2\{60\theta + 5\theta^3 + \theta^5 + 5\theta(12 + \theta^2)\cos\theta - 120\sin\theta\}}{5\{h^5(-1 + e^h) - \theta^5 + \theta^5 \cos\theta\}}\right] l_i \\ & + 12\left(\frac{m_{i-1} + m_i}{h^2}\right) - (M_{i-1} + M_i) + 24\left(\frac{y_{i-1} - y_i}{h^3}\right) \\ & = 0, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{2.6}$$