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ORIGINAL ARTICLE

Convergence to non-autonomous differential equations of second order

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Abstract In this paper, we consider a non-autonomous differential equation of second order $x'' + m(t)f(x') + n(t)g(x) = p(t, x, x')$.

By the Lyapunov function approach, we discuss the convergence of solutions of the equation considered. Our findings generalize some earlier results in the literature.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 34D20; 34C11; 34D05

1. Introduction

In the literature, the qualitative behaviors of solutions of nonlinear differential equations of second order have been investigated by several authors. In this sense, significant results on the convergence of solutions to nonlinear differential equations of second order have been obtained, see Cartwright [1], Ezeilo [2–4], Jin [5], Loud [6], Reissig et al. [7], and Yoshizawa [8,9]. It should be noted that Cartwright [1], proved that if g'' exists and both f and g' are strictly positive for all x, then all ultimately bounded solutions of differential equation $x'' + f(x)x' + g(x) = e(t)$ converge provided that $g(0) = 0$ and $|g''(x)|$ is sufficiently small. In the special case f = constant, the restriction on g'' is not always needed for the convergence of solutions of the mentioned equation. Later, Loud [6] considered the second order differential equation $x'' + cx' + g(x) = e(t)$, where $c > 0$ is a constant. The author showed that if g' exists and satisfies $g' \geqslant b > 0$ for all x, then all solutions $x(t)$ of the equation which ultimately lie in the range $|x| \leqslant A$ are convergent provided that $\max g'(x) < \frac{1}{2}c^2$. Later on, in [2], Ezeilo concerned with the convergence result of Loud [6]. He showed that the result is true even when g is non-differentiable so long as the incrementary ratio $0 < b \leqslant \frac{g(x_2)-g(x_1)}{x_2-x_1} < c^2 (x_2 \neq x_1)$ satisfies.

In this paper, we consider the non-autonomous differential equation of second order

$$x'' + m(t)f(x') + n(t)g(x) = p(t, x, x'), \tag{1}$$

where the functions m, n, f, g and p are real valued and continuous in their respective arguments such that Routh Hurwitz conditions and the uniqueness theorem is valid. It is also

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assumed that the solutions are continuously dependent on the initial conditions and function $p(t, x, x')$ has the form

$$p(t, x, x') = q(t) + r(t, x, x')$$

with the functions q and r depending explicitly on the arguments displayed and are continuous in their respective arguments. Furthermore, it is assumed that $r(t, 0, 0) = 0$ for all t and $f(0) = g(0) = 0$.

The motivation for this work has been inspired basically by the paper of Ezeilo[2] and those listed above. Throughout all the papers and books listed above Lyapunov functios are used to verify the convergence results established there. Our aim here is to extend the results established by Ezeilo [2] to a non-autonomous differential equation of second order, Eq. (1), for the convergence of the solutions. It should be noted that to the best of our knowledge, we did not find any work in the literature based on the results of Ezeilo [2]. Perhaps, the possible difficulties raise this case is due to construction or definition of a suitable Lyapunov function that yielded any meaningful result about the convergence of the solutions for more general differential equations of second order than those in [2]. Further, it is well known that the differential equations of second order have a great scientific importance in many scientific areas. Here, we would not like to give the details of applications. Therefore, investigating the qualitative behavior of the solutions of these type equations is very considerable. It is also worth mentioning that our paper is a continuation of the mentioned works in [1–9]. To achieve our goal, we define a suitable Lyapunov function for Eq. (1) and obtain sufficient conditions to guarantee the convergence of solutions of Eq. (1). Our results are different from that obtained in the literature, (see [1–9] and the references thereof). Namely, the equation considered and the assumptions established here are different from those in the mentioned works above. It should be noted that this paper has also a contribution to the subject in the literature, and it may be useful for researchers working on the qualitative behaviors of solutions to non-autonomous differential equations of the second order. This case is important for the novelty of this paper and an improvement on the topic for the literature.

Definition 1. Any two solutions $x_1(t), x_2(t)$ of Eq. (1) are said to converge (to each other) if $x_1 - x_2 \to 0, x_1' - x_2' \to 0$ as $t \to \infty$.

2. Main results

Theorem 1. *We assume that there are positive constants* $m_1, m_0, n_1, n_0, a, a_0, b$ *and* b_0 *such that*

$$m_0 \leqslant m(t) \leqslant m_1, n_0 \leqslant n(t) \leqslant n_1,$$

$$b \leqslant \frac{n(t)[g(x_2) - g(x_1)]}{x_2 - x_1} \leqslant b_0, \quad (x_2 \neq x_1), \tag{2}$$

$$a \leqslant \frac{m(t)[f(y_2) - f(y_1)]}{y_2 - y_1} \leqslant a_0, \quad (y_2 \neq y_1),$$

and there is a continuous function $\phi(t)$ *such that*

$$|r(t, x_2, y_2) - r(t, x_1, y_1)| \leqslant \phi(t)\{|x_2 - x_1| + |y_2 - y_1|\}, \tag{3}$$

for arbitrary t, x_1, y_1, x_2, y_2 *and*

$$\int_0^t \phi^v(\tau)d\tau \leqslant D_1 t \tag{4}$$

for some a constant $D_1 > 0$*, where* $1 \leqslant v \leqslant 2$*. Then all solutions of Eq. (1) converge.*

Theorem 2. *Let* $x_1(t), x_2(t)$ *be any two solutions of Eq. (1). Suppose that all the conditions of Theorem 1 hold. Then, for each fixed* v *in the range* $1 \leqslant v \leqslant 2$*, there exist constants* D_2, D_3*, and* D_4 *such that*

$$S(t_2) \leqslant D_2 S(t_1) \exp\left\{-D_3(t_2 - t_1) + D_4 \int_{t_1}^{t_2} \phi^v(\tau)d\tau\right\}$$

$$\text{for } t_2 \geqslant t_1, \tag{5}$$

where

$$S(t) = [x_2(t) - x_1(t)]^2 + [x_2'(t) - x_1'(t)]^2. \tag{6}$$

We have the following corollaries, when $x_1(t) = 0$ *and* $t_1 = 0$.

Corollary 1. *Suppose that* $p = 0$ *in Eq. (1) and the assumptions of Theorem 1 hold for arbitrary* $\eta \neq 0$*. Then the trivial solution of Eq. (1) is exponentially stable.*

Corollary 2. *If* $p \neq 0$ *and the assumptions of Theorem 1 hold for arbitrary* $\eta(\eta \neq 0)$*, and* $\xi = 0$*, then there exists a constant* $D_5 > 0$ *such that every solution* $x(t)$ *of Eq. (1) satisfies*

$$|x(t)| \leqslant D_5, |x'(t)| \leqslant D_5.$$

Proof of Theorem 2. Writing Eq. (1) as a system of first order equations, we obtain

$$x' = y, \tag{7}$$
$$y' = -m(t)f(y) - n(t)g(x) + r(t, x, y) + q(t).$$

Let $(x_i(t), y_i(t)), (i = 1, 2)$, be two solutions of system (7).

For the proof of the convergence theorem, we define a function

$$2V = \frac{\delta}{ab}[a^2 + b(b+1)]x^2 + \frac{\delta}{ab}(b+1)y^2 + \frac{2\delta}{b}xy, \tag{8}$$

where a, b, δ are positive real numbers. Indeed, we can rearrange the function in (8) as the following:

$$2V = \frac{\delta}{ab}(y + ax)^2 + \frac{\delta}{a}(b+1)x^2 + \frac{\delta}{a}y^2. \tag{9}$$

Clearly, the function V is positive definite. We can therefore find a constant $D_6 > 0$ such that

$$D_6(x^2 + y^2) \leqslant V, \tag{10}$$

where $D_6 = \frac{1}{2}\min\left\{\frac{\delta}{a}(b+1), \frac{\delta}{a}\right\}$.

Furthermore, by using the estimate $|x||y| \leqslant \frac{1}{2}(y^2 + x^2)$, we can get a constant $D_7 > 0$ such that

$$V \leqslant D_7(x^2 + y^2), \tag{11}$$

where $D_7 = \frac{\delta}{2ab}\max\{a(a+1) + b(b+1), a + b + 1\}$.

Using the estimates (10) and (11), we have