Journal of the Egyptian Mathematical Society (2015) **23**, 67–72

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

On asymptotically ideal equivalent sequences

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Received 16 September 2013; revised 13 December 2013; accepted 27 January 2014
Available online 18 March 2014

KEYWORDS

Ideal;
I-convergence;
Asymptotically equivalent sequence

Abstract In this article we introduce the notion of asymptotically I-equivalent sequences. We prove the decomposition theorem for asymptotically I-equivalent sequences. Further, we will present four theorems that characterize asymptotically I-equivalent of multiple λ and the regularity of asymptotically I-convergence by using a sequence of infinite matrices.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 40A05; 40A99; 40C15

1. Introduction

Throughout $w, \ell_\infty, c, c_0, c^I, c_0^I, m^I$, and m_0^I denote *all, bounded, convergent, null, I-convergent, I-null, bounded I-convergent* and *bounded I-null* class of sequences, respectively. Also $\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and set of real numbers, respectively. Further S_0^I denote the subset of the space m_0^I with non-zero terms.

The notion of statistical convergence is a very useful functional tool for studying the convergence problems of numerical sequences/matrices (double sequences) through the concept of density. It was first introduced by Fast [1] and Schoenberg [2], independently for the real sequences. Later on it was further investigated from sequence space point of view and linked with the summability theory by Fridy [3] and many others. The idea is based on the notion of natural density of subsets of $\mathbb{N}$, the set of positive integers, which is defined as follows. The natural density of a subset of $\mathbb{N}$ is denoted by $\delta(E)$ and is defined by

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} |\{k \leqslant n : k \in E\}|,$$

where the vertical bar denotes the cardinality of the respective set.

The notion of I-convergence (I denotes an ideal of subsets of $\mathbb{N}$), which is a generalization of statistical convergence, was introduced by Kostyrko et al. [4]. Later on it was further investigated from sequence space point of view and linked with summability theory by Šalát et al. [5,6], Tripathy and Hazarika [7] and many others.

A non-empty family of sets $I \subseteq P(\mathbb{N})$ (power set of $\mathbb{N}$) is called an ideal of $\mathbb{N}$ if (i) for each $A, B \in I$, we have $A \cup B \in I$; (ii) for each $A \in I$ and $B \subseteq A$, we have $B \in I$. A family $F \subseteq P(\mathbb{N})$ (power set of $\mathbb{N}$) is called a filter of $\mathbb{N}$ if (i) $\phi \notin F$; (ii) for each $A, B \in F$, we have $A \cap B \in F$; and (iii) for each $A \in F$ and $B \supset A$, we have $B \in F$. An ideal I is called non-trivial if $I \neq \phi$ and $\mathbb{N} \notin I$. It is clear that $I \subseteq P(\mathbb{N})$ is a non-trivial ideal if and only if the class $F = F(I) = \{\mathbb{N} - A : A \in I\}$ is a filter on $\mathbb{N}$. The filter $F(I)$ is called the filter associated with the ideal I. A non-trivial ideal $I \subseteq P(\mathbb{N})$ is called an admissible

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Peer review under responsibility of Egyptian Mathematical Society.

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http://dx.doi.org/10.1016/j.joems.2014.01.011

ideal of $\mathbb{N}$ if it contains all singletons, i.e., if it contains $\{\{x\} : x \in \mathbb{N}\}$.

In [8], Marouf introduced the definition for asymptotically equivalent of two sequences. In [9], Pobyvancts introduced the concept of asymptotically regular matrices, which preserve the asymptotic equivalence of two nonnegative numbers sequences. The frequent occurrence of terms having zero value makes a term-by-term ratio inapplicable in many cases. In [3], Fridy introduced new ways of comparing rates of convergence. If x is in ℓ^1 , he used the remainder sum, whose nth term is $R_n(x) := \sum_{k=n}^{\infty}|x_k|$, and examined the ratio $\frac{R_n(x)}{R_n(y)}$ as $n \to \infty$. If x is a bounded sequence, he used the supremum of the remaining terms which is given by $\mu_n x := \sup_{k\geqslant n}|x_k|$. In [10], Patterson introduced the concept of asymptotically statistically equivalent sequences and natural regularity conditions for nonnegative summability matrices.

In present study we introduce the definition of asymptotically I-equivalent sequences and prove the decomposition theorem for asymptotically I-equivalent sequences and some interesting theorems related to this notion.

2. Definitions and notations

Definition 2.1. [1,3]. A sequence (x_k) is said to be textitstatistically convergent to x_0 if for each $\varepsilon > 0$, the set $E(\varepsilon) = \{k \in \mathbb{N} : |x_k - x_0| \geqslant \varepsilon\}$ has natural density zero.

Definition 2.2 [4]. A sequence (x_k) is said to be *I-convergent* if there exists a number x_0 such that for each $\varepsilon > 0$, the set

$$\{k \in \mathbb{N} : |x_k - x_0| \geqslant \varepsilon\} \in I.$$

Definition 2.3 [4]. Let (x_k) and (y_k) be two real sequences, then we say that $x_k = y_k$for almost all k related to I ($a.a.k.r.I$) if the set $\{k \in \mathbb{N} : x_k \neq y_k\}$ belongs to I.

Definition 2.4 [4]. An admissible ideal I is said to have the property (AP) if for any sequence $\{A_1, A_2, \ldots\}$ of mutually disjoint sets of I, there is sequence $\{B_1, B_2, \ldots\}$ of sets such that each symmetric difference $A_i \Delta B_i (i = 1, 2, 3, \ldots)$ is finite and $\bigcup_{i=1}^{\infty} B_i \in I$.

Example 2.1. If we take $I = I_f = \{A \subseteq \mathbb{N} : A \text{ is a finite subset}\}$. Then, I_f is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincide with usual convergence of sequences.

Example 2.2. If we take $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$, where $\delta(A)$ denote the asymptotic density of the set A. Then I_δ is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincide with statistical convergence of sequences.

Let $\ell^1 = \{x = (x_k) : \sum_{k=1}^{\infty}|x_k| < \infty\}$.

For a summability transformation A, we use $d(A)$ to denote the domain of A:

$$d(A) = \left\{ x = (x_k) : \lim_n \sum_{k=1}^{\infty} a_{n,k} x_k \text{ exists} \right\}.$$

Also $S_\delta = \{x = (x_k) : x_k \geqslant \delta > 0 \text{ for all } k\}$ and

S_0 = {the set of all nonnegative sequences which have at most a finite number of zero entries}.

For a sequence $x = (x_k)$ in ℓ^1 or ℓ_∞, we also define

$$R_n(x) := \sum_{k=n}^{\infty}|x_k| \text{ and } \mu_n x := \sup_{k\geqslant n}|x_k| \text{ for } n \geqslant 0.$$

Definition 2.5 [8]. Two nonnegative sequences (x_k) and (y_k) are said to be *asymptotically equivalent*, written as $x \sim y$ if $\lim_k \frac{x_k}{y_k} = 1$.

Definition 2.6. If $A = (a_{n,k})$ is a sequence of infinite matrices, then a sequence $x = (x_k) \in \ell_\infty$ is said to be *A-summable* to the value x_0 if

$$\lim_n (Ax)_n = \lim_n \sum_{k=1}^{\infty} a_{n,k} x_k = x_0.$$

Definition 2.7. A summability matrix A is *asymptotically regular* provided that $Ax \sim Ay$ whenever $x \sim y, x \in S_0$ and $y \in S_\delta$ for some $\delta > 0$.

The following results will be used for establishing some results of this article.

Lemma 2.1 (*Pobyvancts [9]*). *A nonnegative matrix A is asymptotically regular if and only if for each fixed integer m,*

$$\lim_{n\to\infty} \frac{a_{n,m}}{\sum_{k=1}^{\infty} a_{n,k}} = 0.$$

Lemma 2.2. *A matrix A which maps c_0 to c_0 if and only if*

(a) $\lim_{n\to\infty} a_{n,k}$ *for* $k = 1, 2, 3, \ldots$
(b) *There exists a number $M > 0$ such that for each $n, \sum_{k=1}^{\infty}|a_{n,k}| < M$. Throughout the article I is an admissible ideal of subsets of $\mathbb{N}$.*

3. Asymptotically *I*-equivalent sequences

In this section we introduce the following definitions and prove the decomposition theorem and some interesting theorems.

Definition 3.1. Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be *asymptotically I-equivalent* of multiple $\lambda \in \mathbb{R}$, written as $x \overset{I_\lambda}{\sim} y$, provided for every $\varepsilon > 0$, and $y_k \neq 0$, the set

$$\left\{ k \in \mathbb{N} : \left| \frac{x_k}{y_k} - \lambda \right| \geqslant \varepsilon \right\}$$

belongs to I and in this case we write $I - \lim_k \frac{x_k}{y_k} = \lambda$, simply asymptotically I-equivalent if $\lambda = 1$. It is easy to observe that $x \overset{I_\lambda}{\sim} y$ is equivalent to $\frac{x_k}{\lambda} \overset{I}{\sim} y_k$. From this observation it follow, that we obtain the same notion if we use all real $\lambda's$, some $\lambda \neq 0$, or just $\lambda = 1$.

Example 3.1. Let us consider the sequences $x = (x_k)$ and $y = (y_k)$ as follows: