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ORIGINAL ARTICLE

Generalized sequence spaces defined by a sequence of moduli

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Abstract In the present paper we introduce the sequence spaces defined by a sequence of modulus function $F = (f_k)$. We study some topological properties and inclusion relations between these spaces.

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1. Introduction and preliminaries

Mursaleen and Noman [1] introduced the notion of λ-convergent and λ-bounded sequences as follows :

Let $\lambda = (\lambda_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive real numbers tending to infinity i.e.

$$0 < \lambda_0 < \lambda_1 < \cdots \text{ and } \lambda_k \to \infty \text{ as } k \to \infty$$

and said that a sequence $x = (x_k) \in w$ is λ-convergent to the number L, called the λ-limit of x if $\Lambda_m(x) \longrightarrow L$ as $m \to \infty$, where

$$\lambda_m(x) = \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) x_k.$$

The sequence $x = (x_k) \in w$ is λ-bounded if $\sup_m |\Lambda_m(x)| < \infty$. It is well known [1] that if $\lim_m x_m = a$ in the ordinary sense of convergence, then

$$\lim_m \left(\frac{1}{\lambda_m} \left(\sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) |x_k - a| \right) \right) = 0.$$

This implies that

$$\lim_m |\Lambda_m(x) - a| = \lim_m \left| \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1})(x_k - a) \right| = 0$$

which yields that $\lim_m \Lambda_m(x) = a$ and hence $x = (x_k) \in w$ is λ-convergent to a.

Let w be the set of all sequences, real or complex numbers and l_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_k)$, normed by $\|x\| = \sup_k |x_k|$, where $k \in \mathbb{N}$, the set of positive integers.

A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

(1) $f(x) = 0$ if and only if $x = 0$,
(2) $f(x + y) \leqslant f(x) + f(y)$ for all $x \geqslant 0, y \geqslant 0$,

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(3) f is increasing
(4) f is continuous from right at 0.

It follows that $f$must be continuous everywhere on $[0,\infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x)=\frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x)=x^p$, $0<p<1$, then the modulus $f(x)$ is unbounded. Subsequently, modulus function has been discussed in ([2–15]) and many others.

Let X be a linear metric space. A function $p: X\to\mathbb{R}$ is called paranorm, if

(1) $p(x)\geqslant 0$, for all $x\in X$,
(2) $p(-x)=p(x)$, for all $x\in X$,
(3) $p(x+y)\leqslant p(x)+p(y)$, for all $x,y\in X$,
(4) if (λ_n) is a sequence of scalars with $\lambda_n\to\lambda$ *as* $n\to\infty$ and (x_n) is a sequence of vectors with $p(x_n-x)\to 0$ *as* $n\to\infty$, then $p(\lambda_n x_n-\lambda x)\to 0$ *as* $n\to\infty$.

A paranorm p for which $p(x)=0$ implies $x=0$ is called total paranorm and the pair (X,p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [16], Theorem 10.4.2, P-183).

Let $F=(f_k)$ be a sequence of modulus function, X be a locally convex Hausdorff topological linear spaces whose topology is determined by a set Q of continuous seminorm q, $p=(p_k)$ be a bounded sequence of positive real numbers. By $w(X)$ be denotes the spaces of all sequences defined over X. Now, we define the following sequence spaces in the present paper:

$$w(\Lambda,F,p,q)=\left\{x\in w(X):\frac{1}{n}\sum_{k=1}^{n}[f_k(q(\Lambda_k(x)-L))]^{p_k}\to 0,\ \text{as } n\to\infty \text{ for some } L\right\},$$

$$w_0(\Lambda,F,p,q)=\left\{x\in w(X):\frac{1}{n}\sum_{k=1}^{n}[f_k(q(\Lambda_k(x)))]^{p_k}\to 0,\ \text{as } n\to\infty\right\}$$

and

$$w_\infty(\Lambda,F,p,q)=\left\{x\in w(X):\sup_n\frac{1}{n}\sum_{k=1}^{n}[f_k(q(\Lambda_k(x)))]^{p_k}<\infty\right\}.$$

If $F(x)=x$, we have

$$w(\Lambda,p,q)=\left\{x\in w(X):\frac{1}{n}\sum_{k=1}^{n}(q(\Lambda_k(x)-L))^{p_k}\to 0,\ \text{as } n\to\infty \text{ for some } L\right\},$$

$$w_0(\Lambda,p,q)=\left\{x\in w(X):\frac{1}{n}\sum_{k=1}^{n}(q(\Lambda_k(x)))^{p_k}\to 0,\ \text{as } n\to\infty\right\}$$

and

$$w_\infty(\Lambda,p,q)=\left\{x\in w(X):\sup_n\frac{1}{n}\sum_{k=1}^{n}(q(\Lambda_k(x)))^{p_k}<\infty\right\}.$$

If $p=(p_k)=1$, for all $k\in\mathbb{N}$, we shall write above spaces as

$$w(\Lambda,F,q)=\left\{x\in w(X):\frac{1}{n}\sum_{k=1}^{n}f_k(q(\Lambda_k(x)-L))\to 0,\ \text{as } n\to\infty \text{ for some } L\right\},$$

$$w_0(\Lambda,F,q)=\left\{x\in w(X):\frac{1}{n}\sum_{k=1}^{n}f_k(q(\Lambda_k(x)))\to 0,\ \text{as } n\to\infty\right\}$$

and

$$w_\infty(\Lambda,F,q)=\left\{x\in w(X):\sup_n\frac{1}{n}\sum_{k=1}^{n}f_k(q(\Lambda_k(x)))<\infty\right\}.$$

The following inequality will be used throughout the paper. If $0<h=\inf p_k\leqslant p_k\leqslant\sup p_k=H$, $D=\max(1,2^{H-1})$ then

$$|a_k+b_k|^{p_k}\leqslant D\{|a_k|^{p_k}+|b_k|^{p_k}\} \tag{1.1}$$

for all k and a_k, $b_k\in\mathbb{C}$. Also $|a|^{p_k}\leqslant\max(1,|a|^H)$ for all $a\in\mathbb{C}$.

The main purpose of this paper is to introduce the sequence spaces defined by a sequence of modulus function $F=(f_k)$. We study some topological properties and prove some inclusion relations between these spaces.

2. Main results

Theorem 2.1. *Let $F=(f_k)$ be a sequence of modulus function, $p=(p_k)$ be a bounded sequence of positive real numbers. Then $w(\Lambda,F,p,q)$, $w_0(\Lambda,F,p,q)$ and $w_\infty(\Lambda,F,p,q)$ are linear spaces over the field of complex numbers $\mathbb{C}$.*

Proof. Let x, $y\in w_0(\Lambda,F,p,q)$ and α, $\beta\in\mathbb{C}$, there exists M_α and N_β integers such that $|\alpha|\leqslant M_\alpha$ and $|\beta|\leqslant N_\beta$. Since F is subadditive and q is a seminorm. Therefore

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}[f_k(q(\Lambda_k(\alpha x+\beta y)))]^{p_k}&\leqslant\frac{1}{n}\sum_{k=1}^{n}[(f_k|\alpha|q(\Lambda_k(x))+f_k|\beta|q(\Lambda_k(y)))]^{p_k}\\&\leqslant D(M_\alpha)^H\frac{1}{n}\sum_{k=1}^{n}[f_k(q(\Lambda_k(x)))]^{p_k}\\&\quad+D(N_\beta)^H\frac{1}{n}\sum_{k=1}^{n}[f_k(q(\Lambda_k(y)))]^{p_k}\to 0.\end{aligned}$$

This proves that $w_0(\Lambda,F,p,q)$ is a linear space. Similarly, we can prove that $w(\Lambda,F,p,q)$ and $w_\infty(\Lambda,F,p,q)$ are linear spaces. □

Theorem 2.2. *Let $F=(f_k)$ be a sequence of modulus function, $p=(p_k)$ be a bounded sequence of positive real numbers. Then $w_0(\Lambda,F,p,q)$ is a paranormed space with paranorm*

$$g(x)=\sup_n\left\{\frac{1}{n}\sum_{k=1}^{n}[f_k(q(\Lambda_k(x)))]^{p_k}\right\}^{\frac{1}{M}},$$

where $H=\sup p_k<\infty$ and $M=\max(1,H)$.

Proof. Clearly, $g(x)=g(-x)$, $x=\theta$ implies $\Lambda_k(x)=\theta$ and such that $q(\theta)=0$ and $f_k(0)=0$, where θ is the zero sequence. Therefore $g(\theta)=0$. Since $p_k/M\leqslant 1$ and $M\geqslant 1$, using the Minkowski's inequality and definition of $F=(f_k)$ for each n,