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ORIGINAL ARTICLE

Exact solutions of nonlinear wave equations using $(G'/G, 1/G)$-expansion method

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Abstract In this paper, the $(G'/G, 1/G)$-expansion method with the aid of Maple is used to obtain new exact travelling wave solutions of the Hamiltonian amplitude equation and the Broer–Kaup equations arise in the analysis of various problems in fluid mechanics, theoretical physics. The travelling wave solutions are expressed by the hyperbolic functions, the trigonometric functions and the rational functions. The method demonstrates power, reliability and efficiency. The method also presents a wider applicability for handling nonlinear wave equations.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 34A34; 35C07; 35C08; 35B10; 35C09

1. Introduction

In the recent years, the exact solutions of nonlinear partial differential equations (NLPDEs) have been investigated by many authors who are interested in nonlinear phenomena which exist in many fields, such as fluid mechanics, chemical physics, chemical kinematics, plasma physics, elastic media, optical fibres, solid state physics, biology, atmospheric phenomena and so on. On the other hand, integrability of nonlinear partial differential equations has been studied [1–10] too. Due to the nonlinearity of differential equations and the high dimension of space variables, it is a difficult job for us to determine whatever exact solutions to nonlinear PDEs. In recent years, with the development of symbolic computation packages like Maple, which enable us to perform the complex computation on computer so different methods for finding exact solutions to nonlinear evolution equations have been proposed, developed and extended. Such as Jacobi elliptic function method [11], Hirota bilinear transformation [12], Darboux and Backlund transform [13], the theta function method [14], symmetry method [15,16], the homogeneous balance method [17,18], sine/cosine method [19–21], Exp function method [22–24], the inverse scattering method [25–27], tanh–coth method [28,29], first integral method [30–33], functional variable method [34], simplest equation method [35,36], (G'/G)-expansion method [37–39].

In the present paper, we will use the two-variable $(G'/G, 1/G)$-expansion method, which can be considered as a generalization of the original (G'/G)-expansion method. As a

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pioneer work, Li et al. [40] has applied the two-variable $(G'/G, 1/G)$-expansion method and found the exact solutions of Zakharov equations. Then Zayed et al. [41,42] determined exact solutions of some nonlinear evolution equations.

The present paper investigates the applicability and effectiveness of the $(G'/G, 1/G)$-expansion method on nonlinear evolution equations and systems of NEEs which model many physical and engineering problems such as wave phenomena in fluids, pulses in biological chains, currents in electrical networks, particle vibrations in lattices, chemical reactions, and optical fibres. In Section 2, we describe this method for finding exact travelling wave solutions of nonlinear evolution equations. In Section 3, we illustrate this method in detail with the new Hamiltonian amplitude equation and Broer–Kaup equations. Finally, some conclusions are given.

2. The $(G'/G, 1/G)$ expansion method

In this section, we describe the main steps of the $(G'/G, 1/G)$ expansion method for finding travelling wave solutions of nonlinear evolution equations. As preparations, consider the second order LODE

$$G''(\xi) + \lambda G(\xi) = \mu \tag{2.1}$$

and we let

$$\phi = G'/G, \quad \psi = 1/G \tag{2.2}$$

for simplicity here and after. Using (2.1) and (2.2) yields

$$\phi' = -\phi^2 + \mu\psi - \lambda, \quad \psi' = -\phi\psi \tag{2.3}$$

From the three cases of the general solutions of the LODE (2.1), we have:

Case 1 When $\lambda < 0$, the general solutions of the LODE (2.1) is

$$G(\xi) = A_1 \sinh\left(\sqrt{-\lambda}\xi\right) + A_2 \cosh\left(\sqrt{-\lambda}\xi\right) + \frac{\mu}{\lambda}$$

and we have

$$\psi^2 = \frac{-\lambda}{\lambda^2 v + \mu^2}(\phi^2 - 2\mu\psi + \lambda) \tag{2.4}$$

where A_1 and A_2 are two arbitrary constants and $v = A_1^2 - A_2^2$.

Case 2 When $\lambda > 0$, the general solutions of the LODE (2.1) is

$$G(\xi) = A_1 \sin\left(\sqrt{\lambda}\xi\right) + A_2 \cos\left(\sqrt{\lambda}\xi\right) + \frac{\mu}{\lambda}$$

and we have

$$\psi^2 = \frac{\lambda}{\lambda^2 v - \mu^2}(\phi^2 - 2\mu\psi + \lambda) \tag{2.5}$$

where A_1 and A_2 are two arbitrary constants and $v = A_1^2 + A_2^2$.

Case 3 When $\lambda = 0$, the general solutions of the LODE (2.1) is

$$G(\xi) = \frac{\mu}{2}\xi^2 + A_1\xi + A_2$$

and we have

$$\psi^2 = \frac{1}{A_1^2 - 2\mu A_2}(\phi^2 - 2\mu\psi) \tag{2.6}$$

where A_1 and A_2 are two arbitrary constants.

Now consider a nonlinear evolution equation, say in two independent variables x and t,

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0 \tag{2.7}$$

In general, the left-hand side of Eq. (2.7) is a polynomial in u and its various partial derivatives. The main steps of the $(G'/G, 1/G)$ expansion method are

Step 1 By wave transformation $\xi = x - ct$ and with $u(x,t) = u(\xi)$, Eq. (2.7) can be reduced to an ODE on $u(\xi)$ with

$$P(u, -cu', u', c^2u'', -cu', u'', \ldots) = 0 \tag{2.8}$$

Step 2 Suppose that the solution of ODE (2.8) can be expressed by a polynomial in ϕ and ψ as

$$u(\xi) = \sum_{i=0}^{N} a_i\phi^i + \sum_{i=1}^{N} b_i\phi^{i-1}\psi \tag{2.9}$$

where $G = G(\xi)$ satisfies the second order LODE (2.1), $a_i(i = 0, \ldots, N)$, $b_i(i = 1, \ldots, N)$, c, λ and μ are constants to be determined later, and the positive integer N can be determined by using homogeneous balance between the highest order derivatives and the nonlinear terms appearing in ODE (2.8).

Step 3 Substituting (2.9) into Eq. (2.8), using (2.3) and (2.4) (or using (2.3), (2.5) and (2.3), (2.6)) the left-hand side of (2.8) can be converted into a polynomial in ϕ and ψ, in which the degree of ψ is not larger than 1. Equating each coefficient of the polynomial to zero yields a system of algebraic equations in $a_i(i = 0, \ldots, N)$, $b_i(i = 1, \ldots, N)$, c, $\lambda(\lambda < 0)$, μ, A_1 and A_2.

Step 4 Solve the algebraic solutions in the Step 3 with the aid of Maple. Substituting the values of $a_i(i = 0, \ldots, N)$, $b_i(i = 1, \ldots, N)$, c, λ, μ, A_1 and A_2 obtained into (2.9), one can obtain the travelling wave solutions expressed by the hyperbolic functions of Eq. (2.8) (or expressed by trigonometric functions and rational functions).

3. Applications of the $(G'/G, 1/G)$ expansion method

3.1. The new Hamiltonian amplitude equation

We first consider the new Hamiltonian amplitude equation [43]

$$iu_x + u_{tt} + 2\sigma|u|^2u - \varepsilon u_{xt} = 0 \tag{3.1.1}$$

where $\sigma = \pm 1$, $\varepsilon \ll 1$. This is an equation which governs certain instabilities of modulated wave trains, with the additional term $-\varepsilon u_{xt}$ overcoming the ill-posedness of the unstable nonlinear Schrödinger equation. It is a Hamiltonian analogue of the Kuramoto–Sivashinski equation which arises in dissipative systems and is apparently not integrable.

Using the transformation

$$u(x,t) = e^{i\theta}f(\xi), \quad \theta = \alpha x - \beta t, \quad \xi = k(x - st) \tag{3.1.2}$$

the Eq. (3.1.1) is carried to ODE