Journal of the Egyptian Mathematical Society (2015) 23, 95–101

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Fixed point results in ordered metric spaces for rational type expressions with auxiliary functions

Sumit Chandok [a], Binayak S. Choudhury [b], Nikhilesh Metiya [c,*]

[a] Department of Mathematics, Khalsa College of Engineering & Technology (Punjab Technical University), Ranjit Avenue, Amritsar 143001, India
[b] Department of Mathematics, Bengal Engineering and Science University, Shibpur, Howrah 711103, West Bengal, India
[c] Department of Mathematics, Bengal Institute of Technology, Kolkata 700150, West Bengal, India

Received 9 August 2013; revised 19 December 2013; accepted 2 February 2014
Available online 15 March 2014

KEYWORDS

Fixed point;
Rational type generalized contraction mappings;
Ordered metric spaces

Abstract In this paper, we establish some fixed point results for mappings involving (ϕ, ψ)-rational type contractions in the framework of metric spaces endowed with a partial order. These results generalize and extend some known results in the literature. Four illustrative examples are given.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 41A50; 47H10; 54H25

1. Introduction

The Banach's contraction mapping principle is one of the most versatile elementary results of mathematical analysis. It is widely applied in different branches of mathematics and is regarded as the source of metric fixed point theory. There is a vast literature dealing with technical extensions and generalizations of Banach's contraction principle, some instances of these works are in [1–16].

In recent times, fixed point theory has developed rapidly in partially ordered metric spaces, that is, metric spaces endowed with a partial ordering. The theory originated at a relatively later point of time. An early result in this direction was established by Turinici in ordered metrizable uniform spaces [17]. Application of fixed point results in partially ordered metric spaces was made subsequently, for example, by Ran and Reurings [18] to solving matrix equations and by Nieto and Rodriguez-Lopez [19] to obtain solutions of certain partial differential equations with periodic boundary conditions. Recently, many researchers have obtained fixed point, common fixed point results in partially ordered metric spaces, some of which are in [20–33].

The purpose of this paper is to establish some fixed point results satisfying a generalized contraction mapping of rational type in metric spaces endowed with partial order using some auxiliary functions. Four illustrative examples are given.

2. Mathematical preliminaries

In [34], Dass and Gupta proved the following fixed point theorem.

* Corresponding author. Tel.: +91 9732559201.
E-mail addresses: chansok.s@gmail.com, chandhok.sumit@gmail.com (S. Chandok), binayak12@yahoo.co.in, binayak@becs.ac.in (B.S. Choudhury), metiya.nikhilesh@gmail.com (N. Metiya).

Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

1110-256X © 2014 Production and hosting by Elsevier B.V. on behalf of Egyptian Mathematical Society.
http://dx.doi.org/10.1016/j.joems.2014.02.002

Theorem 2.1 [34]. *Let $(X,\ d)$ be a complete metric space and $T: X \to X$ a mapping such that there exist $\alpha,\ \beta \geqslant 0$ with $\alpha + \beta < 1$ satisfying*

$$d(Tx,\ Ty) \leqslant \alpha\, \frac{d(y,\ Ty)\ [1 + d(x,\ Tx)]}{1 + d(x,\ y)} + \beta\, d(x,\ y), \quad \text{for all } x,\ y \in X. \tag{2.1}$$

Then T has a unique fixed point.

In [35], Cabrera, Harjani and Sadarangani proved the above theorem in the context of partially ordered metric spaces.

Definition 2.1. Suppose $(X,\ \leqslant)$ is a partially ordered set and $T: X \to X$. T is said to be *monotone nondecreasing* if for all $x,\ y \in X$,

$$x \leqslant y \ \Rightarrow\ Tx \leqslant Ty. \tag{2.2}$$

Theorem 2.2 [35]. *Let $(X,\ \leqslant)$ be a partially ordered set and suppose that there exists a metric d in X such that $(X,\ d)$ is a complete metric space. Let $T: X \to X$ be a continuous and nondecreasing mapping such that (2.1) is satisfied for all $x,\ y \in X$ with $x \leqslant y$. If there exists $x_0 \in X$ such that $x_0 \leqslant Tx_0$ then T has a fixed point.*

Theorem 2.3 [35]. *Let $(X,\ \leqslant)$ be a partially ordered set and suppose that there exists a metric d in X such that $(X,\ d)$ is a complete metric space. Assume that if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \leqslant x$, for all $n \in \mathbf{N}$. Let $T: X \to X$ be a nondecreasing mapping such that (2.1) is satisfied for all $x,\ y \in X$ with $x \leqslant y$. If there exists $x_0 \in X$ such that $x_0 \leqslant Tx_0$ then T has a fixed point.*

Theorem 2.4 [35]. *In addition to the hypotheses of Theorem 2.2 (or Theorem 2.3), suppose that for every $x,\ y \in X$, there exists $u \in X$ such that $u \leqslant x$ and $u \leqslant y$. Then T has a unique fixed point.*

Khan et al. [36] initiated the use of a control function that alters distance between two points in a metric space, which they called an altering distance function.

Definition 2.2 [36]. A function $\phi : [0, \infty) \to [0, \infty)$ is called an *altering distance function* if the following properties are satisfied:

(i) ϕ is monotone increasing and continuous,
(ii) $\phi(t) = 0$ if and only if $t = 0$.

In our results in the following section we will use the following class of functions.

We denote

$\Phi = \{\phi : [0, \infty) \to [0, \infty) : \phi$ an altering distance function$\}$ and

$$\Psi = \{\psi : [0, \infty) \to [0, \infty) : \text{for any sequence } \{x_n\} \text{ in } [0,\ \infty) \text{ with } x_n \to t > 0,\ \underline{\lim}\ \psi(x_n) > 0\}.$$

We note that Ψ is nonempty. For, we define ψ on $[0, \infty)$ by $\psi(t) = e^t,\ t \in [0, \infty)$. Then $\psi \in \Psi$. Here we observe that $\psi(0) = 1 > 0$. On the other hand, if $\psi(t) = t^2,\ t \in [0, \infty)$, then $\psi \in \Psi$ and $\psi(0) = 0$.

Note: For any $\psi \in \Psi$, it is clear that $\psi(t) > 0$ for $t > 0$; and $\psi(0)$ need not be equal to 0.

3. Main results

Theorem 3.1. *Let $(X,\ \leqslant)$ be a partially ordered set and suppose that there exists a metric d on X such that $(X,\ d)$ is a complete metric space. Let $T: X \to X$ be a continuous and nondecreasing mapping such that for all $x,\ y \in X$ with $x \leqslant y$,*

$$\phi(d(Tx,\ Ty)) \leqslant \phi(M(x,\ y)) - \psi(N(x,\ y)), \tag{3.1}$$

where $\phi \in \Phi,\ \psi \in \Psi$,

$$M(x,\ y) = \max\left\{\frac{d(y,\ Ty)\ [1 + d(x,\ Tx)]}{1 + d(x,\ y)},\ \frac{d(y,\ Tx)\ [1 + d(x,\ Ty)]}{1 + d(x,\ y)},\ d(x,\ y)\right\}$$

and

$$N(x,\ y) = \max\left\{\frac{d(y,\ Ty)\ [1 + d(x,\ Tx)]}{1 + d(x,\ y)},\ d(x,\ y)\right\}.$$

If there exists $x_0 \in X$ with $x_0 \leqslant Tx_0$, then T has a fixed point.

Proof. If $Tx_0 = x_0$, then we have the result. Suppose that $x_0 < Tx_0$. Then we construct a sequence $\{x_n\}$ in X such that

$$x_{n+1} = Tx_n, \ \text{for every } n \geqslant 0. \tag{3.2}$$

Since T is a nondecreasing mapping, we obtain by induction that

$$x_0 < Tx_0 = x_1 \leqslant Tx_1 = x_2 \leqslant \cdots \leqslant Tx_{n-1} = x_n \leqslant Tx_n = x_{n+1} \leqslant \cdots \tag{3.3}$$

If there exists $n \geqslant 1$ such that $x_{n+1} = x_n$, then from (3.2), $x_{n+1} = Tx_n = x_n$, that is, x_n is a fixed point of T and the proof is finished. Suppose that $x_{n+1} \neq x_n$, that is, $d(x_{n+1},\ x_n) \neq 0$, for all $n \geqslant 1$. Let $R_n = d(x_{n+1},\ x_n)$, for all $n \geqslant 0$.

Since $x_{n-1} < x_n$, for all $n \geqslant 1$, from (3.1), we have

$$\begin{aligned}
\phi(d(x_n, x_{n+1})) &= \phi(d(Tx_{n-1}, Tx_n)) \\
&\leqslant \phi\left(\max\left\{\frac{d(x_n, Tx_n)[1 + d(x_{n-1}, Tx_{n-1})]}{1 + d(x_{n-1}, x_n)},\ \frac{d(x_n, Tx_{n-1})[1 + d(x_{n-1}, Tx_n)]}{1 + d(x_{n-1}, x_n)},\ d(x_{n-1}, x_n)\right\}\right) \\
&\quad - \psi\left(\max\left\{\frac{d(x_n, Tx_n)[1 + d(x_{n-1}, Tx_{n-1})]}{1 + d(x_{n-1}, x_n)},\ d(x_{n-1}, x_n)\right\}\right) \\
&= \phi\left(\max\left\{\frac{d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)},\ \frac{d(x_n, x_n)[1 + d(x_{n-1}, x_{n+1})]}{1 + d(x_{n-1}, x_n)},\ d(x_{n-1}, x_n)\right\}\right) \\
&\quad - \psi\left(\max\left\{\frac{d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)},\ d(x_{n-1}, x_n)\right\}\right) \\
&= \phi(\max\{d(x_n, x_{n+1}),\ d(x_{n-1}, x_n)\}) - \psi(\max\{d(x_n, x_{n+1}),\ d(x_{n-1}, x_n)\}),
\end{aligned}$$

that is,

$$\phi(R_n) = \phi(\max\{R_n,\ R_{n-1}\}) - \psi(\max\{R_n,\ R_{n-1}\}). \tag{3.4}$$

If $R_n > R_{n-1}$, then from (3.4), we have

$$\phi(R_n) \leqslant \phi(R_n) - \psi(R_n), \ \text{that is,}\ \psi(R_n) \leqslant 0,$$

which is a contradiction. So, $R_n \leqslant R_{n-1}$, that is, $\{R_n\}$ is a decreasing sequence. Then the inequality (3.4) yields that

$$\phi(R_n) \leqslant \phi(R_{n-1}) - \psi(R_{n-1}). \tag{3.5}$$