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ORIGINAL ARTICLE

Analytical solution of Abel integral equation arising in astrophysics via Laplace transform

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Abstract The main aim of the present work is to propose a new and simple algorithm for Abel integral equation, namely homotopy perturbation transform method (HPTM). The homotopy perturbation transform method is an innovative adjustment in Laplace transform algorithm (LTA) and makes the calculation much simpler. Abel's integral equation occurs in the mathematical modeling of several models in physics, astrophysics, solid mechanics and applied sciences. The numerical solutions obtained by proposed method indicate that the approach is easy to implement and computationally very attractive. Finally, several numerical examples are given to illustrate the accuracy and stability of this method.

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1. Introduction

Abel's equation is one of the integral equations derived directly from a concrete problem of physics, without passing through a differential equation. This integral equation occurs in the mathematical modeling of several models in physics, astrophysics, solid mechanics and applied sciences. The great mathematician Niels Abel, gave the initiative of integral equations in 1823 in his study of mathematical physics [1–4]. In 1924, generalized Abel's integral equation on a finite segment was studied by Zeilon [5]. The different types of Abel integral equation in physics have been solved by Pandey et al. [6], Kumar and Singh [7], Kumar et al. [8], Dixit et al. [9], Yousefi [10], Khan and Gondal [11], Li and Zhao [12] by applying various kinds of analytical and numerical methods.

Let a material point of mass m move under the influence of gravity on a smooth curve lying in a vertical plane. Let the time t which is required for the point to move along the curve from the height x to the lowest point of the curve be a given function f of x, lead to the integral equation

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$$f(x) = \int_0^x \frac{\varphi(t)}{\sqrt{2g(x-t)}} dt, \tag{1.1}$$

where g is the acceleration due to the gravity.

The main aim of this article is to present analytical and approximate solution of Abel's integral equation by using new mathematical tool like homotopy perturbation transform method. The proposed method is coupling of the homotopy perturbation and Laplace transform method. The main advantage of this proposed method is its capability of combining two powerful methods for obtaining rapid convergent series for singular integral equation of Abel type. The homotopy perturbation method was proposed first by the He in 1998 and was further developed and improved by him [13–17]. The HPM has been successfully applied by many researchers for solving integral equation and differential equations [7–9, 18–22]. In recent years, many authors have paid attention to studying the solutions of integral equation and differential equation by using various methods with combined the Laplace transform. Among these are the Laplace decomposition methods [23–25], homotopy perturbation transform method [26–30]. The elegance of this article can be attributed to the simplistic approach in seeking the approximate analytical solution of the problem.

2. Basic idea of newly proposed method for Abel's integral equation

To illustrate the basic idea of the HPTM for solution of singular integral equation of Abel type, we consider the following Abel's integral equation of second kind as

$$y(x) = f(x) + \int_0^x \frac{y(t)}{\sqrt{x-t}} dt, \quad 0 \leqslant x \leqslant 1. \tag{2.1}$$

Operating the Laplace transform on both sides in Eq. (2.1), we get

$$L[y(x)] = L[f(x)] + L\left\{\int_0^x \frac{y(t)}{\sqrt{x-t}} dt\right\}. \tag{2.2}$$

By using the convolution property of the Laplace transform, Eq. (2.2) takes the form

$$L[y(x)] = L[f(x)] + \sqrt{\frac{\pi}{s}} L[y(x)]. \tag{2.3}$$

Operating the inverse Laplace transform on both sides in Eq. (2.3), we get

$$y(x) = f(x) + L^{-1}\left\{\sqrt{\frac{\pi}{s}} L[y(x)]\right\}. \tag{2.4}$$

We seek the solution of the Abel integral Eq. (2.1) in the following series form

$$\psi(x) = \sum_{n=0}^{\infty} p^n \psi_n(x), \tag{2.5}$$

where $\psi_i(x)$, $i = 0, 1, 2, 3, \ldots$. are functions to be determined. We use the following iterative scheme to evaluate $\psi_i(x)$.

To solve Eq. (2.1) by HPTM, we consider the following convex homotopy [13–17]:

$$\sum_{n=0}^{\infty} p^n \psi_n(x) = f(x) + p\left\{L^{-1}\left(\sqrt{\frac{\pi}{s}} L\left(\sum_{n=0}^{\infty} p^n \psi_n(x)\right)\right)\right\}. \tag{2.6}$$

This is coupling of the Laplace transform and homotopy perturbation method. Now, equating the coefficient of corresponding power of p on both sides, the following approximations are obtained as:

$$p^0 : \psi_0(x) = f(x), \quad p^n : \psi_n(x) = L^{-1}\left\{\sqrt{\frac{\pi}{s}} L(\psi_{n-1}(x))\right\},$$
$$n = 1, 2, 3, \ldots \tag{2.7}$$

Hence the solution of the Eq. (2.1) is given as

$$y(x) = \underset{p\to 1}{Lim} \psi(x) = \sum_{n=0}^{\infty} \psi_n(x). \tag{2.8}$$

It is to be noted that the rate of convergence of the series (2.8) depends upon the initial choices $\psi_0(x)$ as illustrated by the given numerical examples. It is worth to note that the major advantage of homotopy perturbation transform method is that the perturbation equation can be freely constructed in many ways (therefore is problem dependent) by homotopy in topology and the initial approximation can also be freely selected.

3. Illustrative examples

In this section we shall illustrate the homotopy perturbation transform technique by several examples. Here all the results are calculated by using the symbolic calculus software Mathematica 7.

Example 1. We consider the following Abel integral equation of the second kind as follows [6]:

$$y(x) = x + \frac{4}{3}x^{3/2} - \int_0^x \frac{y(t)}{\sqrt{x-t}} dt, \quad 0 \leqslant x \leqslant 1, \tag{3.1}$$

with exact solution $y(x) = x$.

By applying the aforesaid homotopy perturbation transform method [13–17], we have

$$\sum_{n=0}^{\infty} p^n \psi_n(x) = x + \frac{4}{3}x^{3/2} - p\left\{L^{-1}\left(\sqrt{\frac{\pi}{s}} L\left(\sum_{n=0}^{\infty} p^n \psi_{n-1}(x)\right)\right)\right\}. \tag{3.2}$$

Now equating the coefficients of corresponding power of p on both sides in Eq. (3.2), the following iterates $\psi_n(x)$, $n = 0, 1, 2, 3, \ldots$ are given as

$$p^0 : \psi_0(x) = x + \frac{4}{3}x^{3/2},$$
$$p^1 : \psi_1(x) = -L^{-1}\left\{\sqrt{\frac{\pi}{s}} L(\psi_0(x))\right\} = -\frac{4x^{3/2}}{3} - \frac{\pi x^2}{2},$$
$$p^2 : \psi_2(x) = -L^{-1}\left\{\sqrt{\frac{\pi}{s}} L(\psi_1(x))\right\} = \frac{\pi x^2}{2} + \frac{8\pi x^{5/2}}{15},$$
$$p^3 : \psi_3(x) = -L^{-1}\left\{\sqrt{\frac{\pi}{s}} L(\psi_2(x))\right\} = -\frac{8\pi x^{5/2}}{15} - \frac{\pi^2 x^3}{6}, \ldots$$
$$p^{25} : \psi_{25}(x) = -L^{-1}\left\{\sqrt{\frac{\pi}{s}} L(\psi_{24}(x))\right\}$$
$$= -\frac{16384\pi^{12}x^{27/2}}{213458046676875} - \frac{\pi^{13}x^{14}}{87178291200}.$$

Finally, we approximate the analytical solution $y(x)$ by the truncated series as