Journal of the Egyptian Mathematical Society (2015) 23, 108–112

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

On successive approximation method for coupled systems of Chandrasekhar quadratic integral equations

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Received 28 December 2013; revised 18 February 2014; accepted 10 March 2014
Available online 3 May 2014

KEYWORDS

Chandrasekhar quadratic integral equation;
Coupled system;
Contraction mapping principle;
Successive approximation method

Abstract In this paper, We study coupled systems of generalized Chandrasekhar quadratic integral equations which has numerous application (see [1–4]) by applying the Contraction mapping principle and successive approximation method.

MATHEMATICAL SUBJECT CLASSIFICATION: Primary 45G10; 45M99; 47H09

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1. Introduction

The quadratic integral equation of the generalized Chandrasekhar's type

$$x(t) = 1 + x(t)\int_0^1 \frac{t\lambda\phi(s)}{t+s}(log(1+|x(s)|))\,ds, \ \ t \in [0,1]$$

was considered in many papers and monographs ([4,5] for instance).

It arose originally in connection with scattering through a homogeneous semi-infinite plane atmosphere [4].

The existence of the well-known Chandrasekar's integral equation was proved under certain assumption that the so-called *characteristic function* ϕ is an even polynomial in s [4]. For such *characteristic function*, it is known that the result solutions can be expressed in terms of Chandrasekar's H-function [4]. This function is immediately related to the angular pattern or single scattering. In astrophysical applications of the Chandrasekar's equation the only restriction, that $\int_0^1 \phi(s)d \leqslant 1/2$ is treated a necessary condition in [5].

Quadratic integral equations are often applicable in the theory of radiative transfer, the kinetic theory of gases, the theory of neutron transport, the queuing theory and the traffic theory. Especially, the so-called quadratic integral equation of Chandrasekhar's type can be very often encountered in many applications [1]. Many authors studied the existence of solutions for several classes of nonlinear quadratic integral equations (see e.g. [6–11]. However, in most of the above literature, the main results are realized with the help of the technique associated with the measure of noncompactness. Instead of using the technique of measure of noncompactness [12] used Schauder fixed point theorem to prove the existence of continuous solutions for some quadratic integral equations.

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Peer review under responsibility of Egyptian Mathematical Society.

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http://dx.doi.org/10.1016/j.joems.2014.03.004

In this paper, we prove the uniqueness of solutions for the coupled system of quadratic integral equation of generalized Chandrasekhar type

$$\begin{aligned} x(t) &= a_1(t) + g_1(t, y(t)) \int_0^1 \frac{t}{t+s} f_1(s, y(s))\, ds, \quad t \in [0,1], \\ y(t) &= a_2(t) + g_2(t, x(t)) \int_0^1 \frac{t}{t+s} f_2(s, x(s))\, ds, \quad t \in [0,1], \end{aligned} \tag{1}$$

Such systems appear in many problems of applied nature, for instance, see ([13–20]). Su [21] discussed a two-point boundary value problem for a coupled system of fractional differential equations. Gafiychuk et al. [20] analyzed the solutions of coupled nonlinear fractional reaction–diffusion equations. The solvability of the coupled systems of integral equations in reflexive Banach space was proved in [22–24].

Let $\mathbb{R}$ be the set of real numbers whereas $I = [0,1]$. Let $L^1 = L^1[0,1]$ be the class of Lebesgue integrable functions on I with the standard norm.

2. Existence theorem

Now, the coupled system (1) will be investigated under the assumptions:

(i) $a_i : I \to R_+ = [0, +\infty),\ i = 1,2$ is continuous on I.
(ii) $f_i, g_i : I \times D \to R_+,\ i = 1,2$ are continuous and there exist positive constants M_i and $N_i, i = 1,2$ such that $|g_i(t,x)| \leqslant M_i$ and $|f_i(t,x)| \leqslant N_i$ for $(t,x) \in I \times D$, where $D \subset R_+$.
(iii) $f_i, g_i, i = 1,2$ satisfy Lipschitz condition with Lipschitz constants L_i and K_i such that,

$$|g_i(t,x) - g_i(t,y)| \leqslant L_i |x - y|,$$
$$|f_i(t,x) - f_i(t,y)| \leqslant K_i |x - y|.$$

Let $C(I)$ be the class of all real functions defined and continuous on I with the norm

$$\|x\| = \max\{|x(t)| : \ t \in I\}.$$

Define the operator T by

$$T(x,y)(t) = (T_1 y(t), T_2 x(t)),$$

where

$$T_1 y(t) = a_1(t) + g_1(t, y(t)) \int_0^1 \frac{t}{t+s} f_1(s, y(s))\, ds, \quad t \in I,$$
$$T_2 x(t) = a_2(t) + g_2(t, x(t)) \int_0^1 \frac{t}{t+s} f_2(s, x(s))\, ds, \quad t \in I,$$

Theorem 1. *Let the assumptions (i)–(iii) be satisfied. Furthermore, if* $\max\{L_i N_i, K_i M_i\} < 1, i = 1,2$. *Then the coupled system (1) has a unique positive solution* $(x,y) \in C(I) \times C(I)$.

Proof. Define

$$U = \{u = (x(t), y(t)) | (x(t), y(t)) \in C(I) \times C(I) : \|(x,y)\|_{C(I)\times C(I)} \leqslant r\}.$$

For $u_1(t) = (x_1(t), y_1(t))$ and $u_2(t) = (x_2(t), y_2(t))$ in U, we deduce that

$$\begin{aligned} |T_1 y_1(t) - T_1 y_2(t)| &\leqslant |g_1(t, y_1(t)) \int_0^1 \frac{t}{t+s} f_1(s, y_1(s))\, ds \\ &\quad - g_1(t, y_2(t)) \int_0^1 \frac{t}{t+s} f_1(s, y_2(s))\, ds| \\ &\leqslant |g_1(t, y_1(t))| \int_0^1 \frac{t}{t+s} |f_1(s, y_1(s)) - f_1(s, y_2(s))|\, ds \\ &\quad + |g_1(t, y_2(t)) - g_1(t, y_1(t))| \int_0^1 \frac{t}{t+s} |f_1(s, y_2(s))|\, ds \\ &\leqslant M_1 K_1 \|y_1 - y_2\| + L_1 N_1 \|y_1 - y_2\|, \end{aligned}$$

then

$$\begin{aligned} \|T_1 y_1(t) - T_1 y_2(t)\| &\leqslant L_1 N_1 \|y_1 - y_2\| + K_1 M_1 \|y_1 - y_2\|. \\ &\leqslant r_1 \|y_1 - y_2\|, \quad r_1 = \max\{L_1 N_1, K_1 M_1\} \end{aligned}$$

By a similar fashion we get

$$\begin{aligned} \|T_2 x_1(t) - T_2 x_2(t)\| &\leqslant L_2 N_2 \|x_1 - x_2\| + K_2 M_2 \|x_1 - x_2\|, \\ &\leqslant r_2 \|x_1 - x_2\|, \quad r_2 = \max\{L_2 N_2, K_2 M_2\}. \end{aligned}$$

Since

$$u_1(t) - u_2(t) = (x_1(t) - x_2(t), y_1(t) - y_2(t)),$$

then

$$\|u_1(t) - u_2(t)\| = \max\{\|x_1(t) - x_2(t)\|, \|y_1(t) - y_2(t)\|\}.$$

Thus

$$\begin{aligned} \|Tu_1(t) - Tu_2(t)\| &= \|(T_1 y_1(t) - T_1 y_2(t), T_2 x_1(t) - T_2 x_2(t))\| \\ &= \max\{\|T_1 y_1(t) - T_1 y_2(t)\|, \|T_2 x_1(t) - T_2 x_2(t)\|\} \\ &\leqslant \max\{r_1 \|y_1 - y_2\|, r_2 \|x_1 - x_2\|\} \\ &\leqslant r \max\{\|y_1 - y_2\|, \|x_1 - x_2\|\},\ r = \max\{r_1, r_2\} \\ &\leqslant r \|u_1(t) - u_2(t)\|. \end{aligned}$$

Hence, we conclude that the coupled system (1) has a unique solution by the *Contraction mapping principle*. This ends the proof. □

3. Method of successive approximations (Picard method)

Theorem 2. *Let the assumptions (i)–(iii) be satisfied. If* $(M_1 K_1 + N_1 L_1)(M_2 K_2 + N_2 L_2) < 1$, *then the coupled system of the quadratic integral equations of Chandrasekar type (1) has a unique positive solution* $(x,y) \in C(I) \times C(I)$.

Proof. It is clear that the operators T_1, T_2 map $C(I)$ into $C(I)$.

Applying Picard method to the coupled system of quadratic integral Eq. (1), the solution is constructed by the sequences

$$\begin{aligned} x_n(t) &= a_1(t) + g_1(t, y_{n-1}(t)) \int_0^1 \frac{t}{t+s} f_1(s, y_{n-1}(s))\, ds,\ n = 1,2,\ldots, \\ x_0(t) &= a_1(t) \\ y_n(t) &= a_2(t) + g_2(t, x_{n-1}(t)) \int_0^1 \frac{t}{t+s} f_2(s, x_{n-1}(s))\, ds,\ n = 1,2,\ldots, \\ y_0(t &= a_2(t). \end{aligned} \tag{2}$$

All the functions $x_n(t)$ and $y_n(t)$ are continuous functions. Also, $x_n(t)$ and $y_n(t)$ can be written as a sum of successive differences:

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