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ORIGINAL ARTICLE

Difference sequence spaces derived by using generalized means

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Abstract This paper deals with new sequence spaces $X(r,s,t;\Delta)$ for $X \in \{l_\infty, c, c_0\}$ defined by using generalized means and difference operator. It is shown that these spaces are complete normed linear spaces and the spaces $X(r,s,t;\Delta)$ for $X \in \{c, c_0\}$ have Schauder basis. Furthermore, the α-, β-, γ-duals of these sequence spaces are computed and also established necessary and sufficient conditions for matrix transformations from $X(r,s,t;\Delta)$ to X.

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1. Introduction

The study of sequence spaces has importance in the several branches of analysis, namely, the structural theory of topological vector spaces, summability theory, Schauder basis theory etc. Besides this, the theory of sequence spaces is a powerful tool for obtaining some topological and geometrical results using Schauder basis.

Let w be the space of all real or complex sequences $x = (x_n)$, $n \in \mathbb{N}_0$. For an infinite matrix A and a sequence space λ, the matrix domain of A is denoted by λ_A and defined as $\lambda_A = \{x \in w : Ax \in \lambda\}$ [1,2]. Basic methods, which are used to determine the topologies, matrix transformations and inclusion relations on sequence spaces can also be applied to study the matrix domain λ_A. In recent times, there is an approach of forming new sequence spaces by using matrix domain of a suitable matrix and characterize the matrix mappings between these sequence spaces.

Kizmaz first introduced and studied the difference sequence spaces in [3]. Later on, several authors including Ahmad and Mursaleen [4], Çolak and Et [5], Başar and Altay [6], Orhan [7], Polat and Altay [8], Aydin and Başar [9], Başar and Altay [10] and others have introduced and studied new sequence spaces defined by using difference operator.

On the other hand, sequence spaces are also defined by using generalized weighted means. Some of them can be viewed in Malkowsky and Savaş [11], Altay and Başar [12]. Mursaleen and Noman [13] introduced a sequence space of generalized means, which includes most of the earlier known sequence spaces. But till 2011, there was no such literature available in which a sequence space is generated by combining both the weighted means and the difference operator. This was

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firstly initiated by Polat et al. [14]. The authors in [14] have introduced the sequence spaces $\lambda(u,v;\Delta)$ for $\lambda \in \{l_\infty, c, c_0\}$ defined as

$$\lambda(u,v;\Delta) = \{x \in w : (G(u,v).\Delta)x \in \lambda\},$$

where $u, v \in w$ such that $u_n, v_n \neq 0$ for all n and the matrices $G(u,v) = (g_{nk}), \Delta = (\delta_{nk})$ are defined by

$$g_{nk} = \begin{cases} u_n v_k & \text{if } 0 \leqslant k \leqslant n, \\ 0 & \text{if } k > n \end{cases}$$

$$\delta_{nk} = \begin{cases} 0 & \text{if } 0 \leqslant k < n-1, \\ (-1)^{n-k} & \text{if } n-1 \leqslant k \leqslant n, \\ 0 & \text{if } k > n, \end{cases}$$

respectively.

The aim of this article is to introduce new sequence spaces defined by using both the generalized means and the difference operator. We investigate some topological properties as well as α-, β-, γ-duals and obtain the bases of the new sequence spaces. Further, we characterize some matrix transformations between these new sequence spaces.

2. Preliminaries

Let l_∞, c and c_0 be the spaces of all bounded, convergent and null sequences $x = (x_n)$ respectively, with norm $\|x\|_\infty = \sup_n |x_n|$. Let bs and cs be the sequence spaces of all bounded and convergent series respectively. We denote by $e = (1, 1, \ldots)$ and e_n for the sequence whose n-th term is 1 and others are zero and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of all natural numbers. A sequence (b_n) in a normed linear space $(X, \|.\|)$ is called a Schauder basis for X if for every $x \in X$ there is a unique sequence of scalars (μ_n) such that

$$\left\| x - \sum_{n=0}^{k} \mu_n b_n \right\| \to 0 \text{ as } k \to \infty,$$

i.e., $x = \sum_{n=0}^{\infty} \mu_n b_n$ [1,15].

For any subsets U and V of w, the multiplier space $M(U,V)$ of U and V is defined as

$$M(U,V) = \{a = (a_n) \in w : au = (a_n u_n) \in V \text{ for all } u \in U\}.$$

In particular,

$$U^\alpha = M(U, l_1),\ \ U^\beta = M(U, cs) \text{ and } U^\gamma = M(U, bs)$$

are called the α-, β- and γ-duals of U respectively [16].

Let $A = (a_{nk})_{n,k}$ be an infinite matrix with real or complex entries a_{nk}. We write A_n as the sequence of the n-th row of A, i.e., $A_n = (a_{nk})_k$ for every n. For $x = (x_n) \in w$, the A-transform of x is defined as the sequence $Ax = ((Ax)_n)$, where

$$A_n(x) = (Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k,$$

provided the series on the right side converges for each n. For any two sequence spaces U and V, we denote by (U,V), the class of all infinite matrices A that map U into V. Therefore $A \in (U,V)$ if and only if $Ax = ((Ax)_n) \in V$ for all $x \in U$. In other words, $A \in (U,V)$ if and only if $A_n \in U^\beta$ for all n [1]. An infinite matrix $T = (t_{nk})_{n,k}$ is said to be triangle if $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0, n \in \mathbb{N}_0$.

3. Sequence space $X(r,s,t;)$ for $X \in \{l_\infty, c, c_0\}$

In this section, we first begin with the notion of generalized means given by Mursaleen et al. [13].

We denote the sets $\mathcal{U}$ and $\mathcal{U}_0$ as

$$\mathcal{U} = \left\{u = (u_n)_{n=0}^{\infty} \in w : u_n \neq 0 \text{ for all } n\right\} \text{ and}$$
$$\mathcal{U}_0 = \left\{u = (u_n)_{n=0}^{\infty} \in w : u_0 \neq 0\right\}.$$

Let $r = (r_n), t = (t_n) \in \mathcal{U}$ and $s = (s_n) \in \mathcal{U}_0$. The sequence $y = (y_n)$ of generalized means of a sequence $x = (x_n)$ is defined by

$$y_n = \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k x_k \qquad (n \in \mathbb{N}_0).$$

The infinite matrix $A(r,s,t)$ of generalized means is defined by

$$(A(r,s,t))_{nk} = \begin{cases} \frac{s_{n-k} t_k}{r_n} & 0 \leqslant k \leqslant n, \\ 0 & k > n. \end{cases}$$

Since $A(r,s,t)$ is a triangle, it has a unique inverse and the inverse is also a triangle [17]. Take $D_0^{(s)} = \frac{1}{s_0}$ and

$$D_n^{(s)} = \frac{1}{s_0^{n+1}} \begin{vmatrix} s_1 & s_0 & 0 & 0 \cdots & 0 \\ s_2 & s_1 & s_0 & 0 \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ s_{n-1} & s_{n-2} & s_{n-3} & s_{n-4} \cdots & s_0 \\ s_n & s_{n-1} & s_{n-2} & s_{n-3} \cdots & s_1 \end{vmatrix} \qquad \text{for } n = 1, 2, 3, \ldots$$

Then the inverse of $A(r,s,t)$ is the triangle $B = (b_{nk})_{n,k}$ which is defined as

$$b_{nk} = \begin{cases} (-1)^{n-k} \frac{D_{n-k}^{(s)}}{t_n} r_k & 0 \leqslant k \leqslant n, \\ 0 & k > n. \end{cases}$$

We now introduce the sequence space $X(r,s,t;\Delta)$ for $X \in \{l_\infty, c, c_0\}$ as

$$X(r,s,t;\Delta) = \left\{ x = (x_n) \in w : \left(\frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k \Delta x_k \right)_n \in X \right\},$$

which is a combination of the generalized means and the difference operator Δ such that $\Delta x_k = x_k - x_{k-1}, x_{-1} = 0$. By using matrix domain, we can write $X(r,s,t;\Delta) = X_{A(r,s,t;\Delta)} = \{x \in w : A(r,s,t;\Delta)x \in X\}$, where $A(r,s,t;\Delta) = A(r,s,t).\Delta$, product of two triangles $A(r,s,t)$ and Δ.

These sequence spaces include many known sequence spaces studied by several authors. For examples,

(i) if $r_n = \frac{1}{u_n}, t_n = v_n$ and $s_n = 1 \forall n$, then the sequence spaces $X(r,s,t;\Delta)$ reduce to $X(u,v;\Delta)$ for $X \in \{l_\infty, c, c_0\}$ introduced and studied by Polat et al. [14],

(ii) if $t_n = 1, s_n = 1\ \forall n$ and $r_n = n+1$, then the sequence space $X(r,s,t;\Delta)$ for $X = l_\infty$ reduces to C_∞ studied by Orhan [7],

(iii) if $r_n = \frac{1}{n!}, t_n = \frac{\alpha^n}{n!}, s_n = \frac{(1-\alpha)^n}{n!}$, where $0 < \alpha < 1$, then the sequence spaces $X(r,s,t;\Delta)$ for $X \in \{l_\infty, c, c_0\}$ reduce to $e_\infty^\alpha(\Delta), e_c^\alpha(\Delta)$ and $e_0^\alpha(\Delta)$ respectively [8],

(iv) if $r_n = n+1, t_n = 1+\alpha^n$, where $0 < \alpha < 1$ and $s_n = 1 \forall n$, then the sequence spaces $X(r,s,t;\Delta)$ for $X \in \{c, c_0\}$ reduce to the spaces of sequences $a_c^\alpha(\Delta)$ and $a_0^\alpha(\Delta)$