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ORIGINAL ARTICLE

On generalized superposition operator acting of analytic function spaces

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Abstract In this paper we introduce a new integration operator $S_{g,\phi}^{(n)}$, where

$$S_{g,\phi}^{(n)} = \int_0^z \phi^{(n)}(f(\xi))g(\xi)d\xi.$$

We characterize all entire functions that transform a Bloch-type space into another by this new integration operator. Also, we prove that all generalized superposition operators induced by such entire functions are bounded.

AMS: 46E15; 47B33; 47B38; 54C35

1. Introduction

Let $H(\mathbb{D})$ denote the space of all analytic functions on the unit disk $\mathbb{D}$ of $\mathbb{C}$. Let ϕ be analytic self-map of $\mathbb{D}$, n be a positive integer and $g \in H(\mathbb{D})$. Let X and Y be two metric spaces of analytic functions on the unit disk and ϕ denotes a complex-valued function of the plan $\mathbb{C}$. The superposition operator S_ϕ on X is defined by

$$S_\phi(f) = \phi \circ f, \quad f \in X.$$

If $S_\phi f \in Y$ for $f \in X$, we say that ϕ acts by superposition from X into Y. We see that if X contains linear functions, ϕ must be an entire function. Let $H(\mathbb{D})$ be the class of all analytic function on $\mathbb{D}$, then for $g \in H(\mathbb{D})$, we define a new nonlinear superposition operator as follows:

$$(S_{g,\phi}^{(n)} f)(z) = \int_0^z \phi^{(n)}(f(\xi))g(\xi)d\xi.$$

The operator $S_{g,\phi}^{(n)}$ is called the generalized superposition operator. When $g = f'$ and $n = 1$, we see that this operator is essentially superposition operator, since the following difference $S_{g,\phi}^{(n)} - S_\phi$ is a constant. Therefore, $S_{g,\phi}^{(n)}$ is a generalization of the superposition operator. To the best of our Knowledge, the operator $S_{g,\phi}^{(n)}$ is introduced in the present paper for the first time. The graph of $S_{g,\phi}^{(n)}$ is usually closed but, since the operator is nonlinear, this is not enough to assure its boundedness. Nonetheless, for a number of important spaces X, Y, such as Hardy, Bergman, Dirichlet, and Bloch, the mere action $S_{g,\phi}^{(n)} : X \to Y$ implies that ϕ must belong to a very special class of entire functions, which in turn implies boundedness.

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Wen Xu studied superposition operators on Bloch-type spaces in [1].

In this paper we give a complete description of the generalized superpositions on Bloch-type spaces in terms of the order and type of ϕ and the degree of polynomials.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Recall that the well known Bloch space (cf. [2]) is defined as follows:

$$\mathcal{B} = \{f : f \text{ analytic in } \mathbb{D} \text{ and } \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty\};$$

the little Bloch space $\mathcal{B}_0$ (cf. [2]) is a subspace of $\mathcal{B}$ consisting of all $f \in \mathcal{B}$ such that

$$\lim_{|z|\to 1^-}(1-|z|^2)|f'(z)| = 0.$$

Definition 1.1 [3]. Let f be an analytic function in $\mathbb{D}$ and $0 < \alpha < \infty$. The α-Bloch space $\mathcal{B}^\alpha$ is defined by

$$\mathcal{B}^\alpha = \{f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| < \infty\},$$

the little α-Bloch space $\mathcal{B}_0^\alpha$ is given as follows

$$\mathcal{B}_0^\alpha = \{f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}_0^\alpha} = \lim_{|z|\to 1^-}(1-|z|^2)^\alpha|f'(z)| = 0\}.$$

The spaces $\mathcal{B}^1$ and $\mathcal{B}_0^1$ are called the Bloch space and denoted by $\mathcal{B}$ and $\mathcal{B}_0$ respectively (see [4]).

As a simple example one can get that the function $f(z) = \log(1-z)$ is a Bloch function but $f(z) = \log^2(1-z)$ is not a Bloch function.

Definition 1.2 (*see [5]*). For $p \in (0,\infty)$ and $-1 < \alpha < \infty$, the Bergman-type spaces $\mathcal{A}_\alpha^p$ are defined by

$$\mathcal{A}_\alpha^p = \{f \in H(\mathbb{D}) : \|f\|_{\mathcal{A}_\alpha^p} = \sup_{z\in\mathbb{D}}|f(z)|^p(1-|z|^2)^\alpha < \infty\}.$$

Moreover, $f \in \mathcal{A}_{0,\alpha}$; if and only if

$$\lim_{|z|\to 1^-}\sup_{z\in\mathbb{D}}|f(z)|(1-|z|^2)^\alpha = 0.$$

Conformally invariant spaces of the disk: It is a standard fact that the set of all disk automorphisms (i.e., of all one-to-one analytic maps φ of $\mathbb{D}$ onto itself\,), denoted $Aut(\mathbb{D})$, coincides with the set of all Möbius transformations of $\mathbb{D}$ onto itself:

$$Aut(\mathbb{D}) = \{\lambda\varphi_a : |\lambda| = 1; a \in \mathbb{D}\},$$

where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ are the automorphisms: $\varphi_a(\varphi_a(z)) \equiv z$.

A space X of analytic functions in $\mathbb{D}$, equipped with a semi-norm ρ, is said to be conformally invariant or Möbius invariant if whenever $f \in X$, then also $f \circ \varphi \in X$ for any $\varphi \in Aut(\mathbb{D})$ and, moreover, $\rho(f \circ \varphi) \leqslant C\rho(f)$ for some positive constant C and all $f \in X$.

Definition 1.3. In topology, a geometrical object or space is called simply connected (or 1-connected) if it is path-connected and every path between two points can be continuously transformed into every other while preserving the two endpoints in question.

Definition 1.4. A path from a point x to a point y in a topological space X is a continuous function f from the unit interval $[0,1]$ to X with $f(0) = x$ and $f(1) = y$. A path-component of X is an equivalence class of X under the equivalence relation defined by x is equivalent to y if there is a path from x to y. The space X is said to be path-connected (or path-wise connected or 0-connected) if there is only one path-component, i.e. if there is a path joining any two points in X.

Remark 1.1. Every path-connected space is connected. The converse is not always true.

In this section, we give some auxiliary results which are incorporated in the following lemmas.

Lemma 1.1. *Let and $f \in \mathcal{B}_\alpha$ and $0 < \alpha < \infty$. Suppose that*

$$I_\alpha = \int_0^1 \frac{|z|dt}{(1-t^2|z|^2)^\alpha} < \infty. \tag{1}$$

Then we have,

$$|f(z)| \leqslant |f(0)| + C\|f\|_{\mathcal{B}^\alpha},$$

for some $C > 0$ independent of f.

Proof. Let $|z| > \frac{1}{2}$, $z = r\xi$, and $\xi \in \partial\mathbb{D}$. We have

$$\left|f(z) - f\left(\frac{r\xi}{2}\right)\right| = \left|\int_{\frac{1}{2}}^1 zf'(tz)dt\right| \leqslant \int_{\frac{1}{2}}^1 |z||f'(tz)|dt$$
$$\leqslant 2\|f\|_{\mathcal{B}^\alpha}\int_0^1 \frac{|z|dt}{(1-t^2|z|^2)^\alpha} \leqslant C\|f\|_{\mathcal{B}^\alpha}.$$

Also, we have

$$|f(z)| \leqslant \max_{|z|\leqslant\frac{1}{2}}|f(z)| + C\|f\|_{\mathcal{B}^\alpha}. \tag{2}$$

Let $|z| \leqslant \frac{1}{2}$, then, by the mean value property of the function $f(z) - f(0)$ (see [6]) and Jensen's inequality, we obtain

$$\max_{|z|\leqslant\frac{1}{2}}|f(z) - f(0)| \leqslant 4^n\int_{|z|\leqslant\frac{3}{4}}|f(w) - f(0)|dA(w)$$
$$\leqslant 4^n\int_{|z|\leqslant\frac{3}{4}}|f'(w)|^2dA(w) \leqslant 3^n\max_{|z|\leqslant\frac{3}{4}}|f'(w)|^2.$$

The second inequality can be easily proved by using the homogeneous expansion of f.

Hence,

$$\max_{|z|\leqslant\frac{1}{2}}|f(z)| \leqslant |f(0)| + (\sqrt{3})^n\max_{|z|\leqslant\frac{3}{4}}|f'(z)|$$
$$\leqslant |f(0)| + \frac{2^{4\alpha}(\sqrt{3})^n}{7^\alpha}\|f\|_{\mathcal{B}^\alpha}. \tag{3}$$

From (2) and (3), the result follows easily when $\alpha \neq 1$. If $\alpha = 1$, then we have

$$|f(z)| \leqslant |f(0)| + \frac{16(\sqrt{3})^n}{7}\|f\|_{\mathcal{B}^1} + C\|f\|_{\mathcal{B}^1}$$
$$\leqslant |f(0)| + \left(\frac{16(\sqrt{3})^n}{7} + C\right)\|f\|_{\mathcal{B}^1}.$$

This complete the proof. □