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ORIGINAL ARTICLE

# Auxiliary problem and algorithm for a generalized mixed equilibrium problem

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**Abstract** In this paper, we consider generalized mixed equilibrium problem and its auxiliary problem in Hilbert space. Further, we establish an existence and uniqueness theorem for the auxiliary problem. Furthermore, using this theorem we construct an algorithm for generalized mixed equilibrium problem and discuss the convergence analysis of the algorithm and existence of solution of generalized mixed equilibrium problem.

**2000 MATHEMATICS SUBJECT CLASSIFICATION:** 49J30; 47H10; 47H17; 90C99



## 1. Introduction

One of the most important and interesting problems in the theories of equilibrium problems and variational inequalities is to develop the methods which give efficient and implementable algorithms for solving equilibrium problems and variational inequalities. These methods include projection method and its variant forms, linear approximation, descent and Newton's methods, and the method based on auxiliary principle technique.

It is well known that the projection method and its variants cannot be extended for mixed equilibrium problems involving non-differentiable term. To overcome this drawback, one uses usually the auxiliary principle technique. This technique deals with finding a suitable auxiliary problem and prove that the solution of an auxiliary problem is the solution of original problem by using fixed-point approach which was used by [1]. Recent work in [2–8], is an extension of this technique to suggest and analyze a number of iterative methods for solving various classes of variational inequalities and equilibrium problems.

Motivated by recent work going in this direction, in this paper, we extend auxiliary principle technique to a generalized mixed equilibrium problem (for short, GMEP) in Hilbert space. We prove existence of the unique solution of an auxiliary problem related to GMEP, which enable us to construct an algorithm for finding the approximate solution of GMEP. Further we prove that the approximate solution is strongly convergent to the unique solution of GMEP. The algorithms and results of this paper are new and different from the algorithms and results of [5]. The results presented here generalize the techniques and results of [2,3].

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## 2. Preliminaries

Let $H$ be a real Hilbert space whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively, and let $K$ be nonempty, closed and convex subset of $H$. Given the single-valued mappings $T, S : H \to H, N, \eta : H \times H \to H$ and a bifunction $f : H \times H \to \mathbb{R}$ such that $f(x,x) = 0 \forall x \in H$, then we consider the *generalized mixed equilibrium problem* (GMEP) of finding $x \in K$ such that

$$f(x,y) + \langle N(Tx, Sx), \eta(y,x) \rangle + b(x,y) - b(x,x) \geqslant 0, \quad \forall y \in K, \tag{2.1}$$

where the bifunction $b : H \times H \to \mathbb{R}$, which is not necessarily differentiable, satisfies the following properties:

(i) $b$ is linear in the first argument;
(ii) $b$ is bounded, that is, there exists a constant $\gamma > 0$ such that $b(x,y) \leqslant \gamma \|x\| \|y\|, \quad \forall x, y \in H$;
(iii) $b(x,y) - b(x,z) \leqslant b(x, y-z), \quad \forall x, y, z \in H$;
(iv) $b$ is convex in the second argument.

*Some special cases:*

(I) If $N(Tx, Sx) = B(x), b(x,y) = 0$ and $\eta(y,x) = y - x \ \forall x, y \in K$, where $B : K \to K$, then GMEP (2.1) reduces to the mixed equilibrium problem of finding $x \in K$ such that

$$f(x,y) + \langle Bx, y - x \rangle \geqslant 0, \quad \forall y \in K, \tag{2.2}$$

which has been studied in [9].

(II) If $f(x,y) = 0; b(x,y) = 0$ and $N(Tx, Sx) = B(x) \ \forall x, y \in K$, where $B : K \to K$, then GMEP (2.1) reduces to the variational-like inequality problem of finding $x \in K$ such that

$$\langle Bx, \eta(y,x) \rangle \geqslant 0, \quad \forall y \in K. \tag{2.3}$$

This problem has been studied in [10].

(III) If $N(Tx, Sx) = 0 \ \forall x \in K$, then GMEP (2.1) reduces to the generalized equilibrium problem of finding $x \in K$ such that

$$f(x,y) + b(x,y) - b(x,x) \geqslant 0, \quad \forall y \in K. \tag{2.4}$$

This problem has been studied in [5].

(IV) If, in (III), $b(x,y) = 0 \ \forall x, y \in K$, then GMEP (2.1) reduces to the equilibrium problem of finding $x \in K$ such that

$$f(x,y) \geqslant 0, \quad \forall y \in K. \tag{2.5}$$

This problem has been studied in [11].

(V) If $N(Tx, Sx) = B(x), b(x,y) = \phi(y) - \phi(x) \ \forall x, y \in K$, where $\phi : K \to \mathbb{R}$ and $f(x,y) = 0 \forall x, y \in K$, then GMEP (2.1) reduces to the problem of finding $x \in K$ such that

$$\langle Bx, \eta(y,x) \rangle + \phi(y) - \phi(x) \geqslant 0, \quad \forall y \in K. \tag{2.6}$$

This problem has been studied in [12] in $\mathbb{R}^n$.

(VI) If, in (V), $\eta(y,x) = y - x \ \forall x, y \in K$, then GMEP (2.1) reduces to the variational inequality problem of finding $x \in K$ such that

$$\langle Bx, y - x \rangle + \phi(y) - \phi(x) \geqslant 0, \quad \forall y \in K. \tag{2.7}$$

This problem has been studied in [13].

## 3. Auxiliary problem and existence of solutions

First related to GMEP (2.1), we consider the auxiliary problem and then establish an existence theorem for the auxiliary problem:

**Auxiliary problem (AP).** Given $x \in K$, find $z \in K$ such that

$$\rho f(z,y) + \langle Az - Ax + \rho N(Tx, Sx), \eta(y,z) \rangle + \rho[b(x,y) - b(x,z)] \geqslant 0, \quad \forall y \in K, \tag{3.1}$$

where $\rho > 0$ is a constant and $A : K \to H$ is not necessarily a linear mapping.

We observe that if $z = x$, clearly $z$ is a solution of GMEP (2.1).

Now, we give the following definitions and concepts.

**Definition 3.1** 14. Let $K$ be a subset of a topological vector space $X$. A set-valued mapping $T : K \to 2^X$ is called *Knaster–Kuratowski–Mazurkiewieg mapping* (KKM mapping), if for each nonempty finite subset $\{x_1, x_2, \ldots, x_n\} \subset K$, we have $\text{Co}\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^n T(x_i)$.

**Lemma 3.1** [14]. *Let $K$ be a subset of a topological vector space $X$ and let $T : K \to 2^X$ be a KKM mapping. If for each $x \in K, T(x)$ is closed and for atleast one $x \in K, T(x)$ is compact, then*

$$\bigcap_{x \in K} T(x) \neq \emptyset.$$

**Definition 3.2.** Let $f : K \times K \to \mathbb{R}; N : H \times H \to H; T, S : K \to K$ and $\eta : H \times H \to H$. Then:

(a) $T$ is said to be *$\alpha$-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$f(x,y) + f(y,x) + \alpha \|x - y\|^2 \leq 0, \quad \forall x, y \in H;$$

(b) $\eta$ is said to be *$\delta$-Lipschitz continuous* if there exists a constant $\delta > 0$ such that

$$\|\eta(x,y)\| \leqslant \delta \|x - y\|, \quad \forall x, y \in H;$$

(c) $A$ is said to be $\tau$-strongly *$\eta$-monotone* if there exists a constant $\tau > 0$ such that

$$\langle Ax - Ay, \eta(x,y) \rangle \geqslant \tau \|x - y\|^2, \quad \forall x, y \in H;$$

(d) $N$ is said to be *$\epsilon$-strongly mixed $\eta$-monotone* with respect to $T$ and $S$, if there exists a constant $\epsilon > 0$ such that

$$\langle N(Tx, Sx) - N(Ty, Sy), \eta(x,y) \rangle \geqslant \epsilon \|x - y\|^2, \quad \forall x, y \in H;$$

(e) $N$ is said to be *$(\beta_1, \beta_2)$-Lipschitz continuous* if there exist constants $\beta_1, \beta_2 > 0$ such that

$$\|N(x_1, y_1) - N(x_2, y_2)\| \leqslant \beta_1 \|x_1 - x_2\| + \beta_2 \|y_1 - y_2\|, \quad \forall x_1, x_2, y_1, y_2 \in H;$$

(f) $f$ and $A$ are said to be *simultaneously hemicontinuous* if for $\lambda \in [0,1], y_\lambda := \lambda y + (1 - \lambda) z, y, z \in K$, we have

$$f(y_\lambda, p) + \langle A(y_\lambda), p \rangle \longrightarrow F(z,p) + \langle A(z), p \rangle$$

as $\lambda \to 0^+$ for any $p \in K$.