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ORIGINAL ARTICLE

On curvatures and points of the translation surfaces in Euclidean 3-space

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Abstract In this paper, translation surfaces generated by two arbitrary space curves in 3-dimensional Euclidean space have been investigated. Furthermore, a classification of some special points on these surfaces have been given. Moreover, we have obtained the associated geometric attributes for these surfaces, e.g. the Gaussian curvature and the mean curvature. Finally, some important results and examples that show the idea were presented.

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1. Introduction

The famous type of a translation surface is generated by two planar curves lying on orthogonal planes. This type will be called as a translation surface of plane type and takes the form:

$$X(u,v) = (u, 0, f(u)) + (0, v, g(v)), \tag{1}$$

where $f(u)$ and $g(v)$ being smooth functions of the variables u and v, respectively.

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The translation surfaces of plane type have been investigated from the various viewpoints by many differential geometers. Verstraelen et al. [1] have investigated minimal translation surfaces of plane type in n-dimensional Euclidean spaces. Liu [2] has given the classification of the translation surfaces of plane type with constant mean curvature or constant Gauss curvature in 3-dimensional Euclidean space $\mathbf{E}^3$ and 3-dimensional Minkowski space $\mathbf{E}^3_1$. Yoon [3] has studied translation surfaces in the 3-dimensional Minkowski space whose Gauss map $\mathbf{G}$ satisfies the condition $\Delta\mathbf{G} = \mathbf{A}\mathbf{G}, \mathbf{A} \in Mat(3,R)$ where Δ denotes the Laplacian of the surface with respect to the induced metric and $Mat(3,R)$ the set of 3×3 real matrices. Dillen et al. [4] have derived a classification of translation surfaces in the 3-dimensional Euclidean and Minkowski space, satisfying the Weingarten condition. Yoon [5] has classified a polynomial translation surfaces in Euclidean 3-space satisfying the Jacobi condition with respect to the Gaussian curvature, the mean curvature and the second Gaussian curvature. Munteanu and Nistor [6] have studied

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the second fundamental form of translation surfaces of plane type in $\mathbf{E}^3$. They have given a non-existence result for polynomial translation surface with vanishing second Gaussian curvature. Bekkar and Senoussi [7] have studied the translation surfaces in the 3-dimensional Euclidean and Lorentz-Minkowski space under the condition $\Delta^{III} r_i = \mu_i r_i, \mu_i \in R$, where Δ^{III} denotes the Laplacian of the surface with respect to the third fundamental form III. They shown that in both spaces a translation surface satisfying the preceding relation is a surface of Scherk. Cetin et al. [8,9] have investigated the translation surfaces according to Frenet frames in Euclidean and Minkowski 3-space. They have given some properties of these surfaces using non-planer space curves.

The general form of translation surface is the surface that can be generated from two arbitrary space curves by translating either of them parallel to itself. In such a way that each of its points describes a curve that is a translation of the other curve. A generalized type of a translation surface parameterized by:

$$X(u,v) = \alpha(u) + \beta(v), \tag{2}$$

where α and β are arbitrary space curves of the parameters u and v (may be the arc-length parameters).

In this paper, we investigated the translation surfaces in 3-dimensional Euclidean space generated by two arbitrary space curves. Furthermore, a classification of flat and minimal translation surfaces has been obtained and some examples for geometric points on these surfaces were introduced.

2. Preliminaries

The Euclidean 3-dimensional space $\mathbf{E}^3$ is provided with the metric given by [10–12]:

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbf{E}^3$. Let $\delta : I \subset \mathbb{R} \to \mathbf{E}^3 : s \mapsto \delta(s)$ be an arbitrary curve in $\mathbf{E}^3$. The curve δ is said to be of unit speed (or parameterized by the arc-length parameter s) if $\langle \delta'(s), \delta'(s) \rangle = 1$ for any $s \in I$. Let $\{t(s), n(s), b(s)\}$ be the moving frame of δ, where the vectors t, n and b are the tangent, normal and binormal vectors, respectively. The Frenet equations for δ are given by

$$\begin{bmatrix} t'(s) \\ n'(s) \\ b'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix}. \tag{3}$$

Let $S : \Phi = \Phi(u,v) \subset \mathbf{E}^3$ be a regular surface. Then the unit normal vector field of the surface S is given by

$$U = \frac{\Phi_u \wedge \Phi_v}{\|\Phi_u \wedge \Phi_v\|}, \quad \Phi_u = \frac{\partial \Phi(u,v)}{\partial u}, \quad \Phi_v = \frac{\partial \Phi(u,v)}{\partial v}, \tag{4}$$

where $\wedge$ stands the vector product of $\mathbf{E}^3$. The first fundamental form of the surface is induced from the metric of the ambient space $\mathbf{E}^3$

$$I = \langle d\Phi, d\Phi \rangle = E\ du^2 + 2F\ du\ dv + G\ dv^2, \tag{5}$$

with coefficients

$$E = \langle \Phi_u, \Phi_u \rangle, \quad F = \langle \Phi_u, \Phi_v \rangle, \quad G = \langle \Phi_v, \Phi_v \rangle.$$

Also, the second fundamental form of the surface S is given by

$$II = -\langle dU, d\Phi \rangle = L\ du^2 + 2M\ du\ dv + N\ dv^2, \tag{6}$$

where

$$L = \langle \Phi_{uu}, U \rangle, \quad M = \langle \Phi_{uv}, U \rangle, \quad N = \langle \Phi_{vv}, U \rangle.$$

Under this parametrization Φ, the Gauss and mean curvatures have the classical expressions, respectively

$$K = \frac{L\ N - M^2}{EG - F^2}, \tag{7}$$

$$H = \frac{E\ N + G\ L - 2F\ M}{2(EG - F^2)}. \tag{8}$$

The principal curvatures of the surface S are defined by

$$k_1 = H + \sqrt{H^2 - K}, \quad k_2 = H - \sqrt{H^2 - K}. \tag{9}$$

In the light of the above, the sectional curvature κ_n and geodesic torsion τ_g are given by

$$\kappa_n = \frac{KH}{2H^2 - K}, \quad \tau_g = \pm \frac{K\sqrt{H^2 - K}}{2[2H^2 - K]}. \tag{10}$$

Now, we can write the following important definition:

Definition 2.1. A regular surface in $\mathbf{E}^3$ is a flat (developable) surface if $K = 0$ and a minimal surface if $H = 0$.

3. Curvatures on the translation surface

Let $X(u,v)$ be a translation surface in Euclidean 3-space $\mathbf{E}^3$ taking the form (2), where the variables u and v are the arc-length parameters for the two generating curves $\alpha(u)$ and $\beta(v)$, respectively. Let $\{t_\alpha,\ n_\alpha,\ b_\alpha\}$ be the Frenet frame field of the curve α with curvature κ_α and torsion τ_α. Also, let $\{t_\beta,\ n_\beta,\ b_\beta\}$ be the Frenet frame field of the curve β with curvature κ_β and torsion τ_β.

Calculating the partial derivative of (2) with respect to u and v respectively, we get

$$X_u = t_\alpha, \quad X_v = t_\beta, \tag{11}$$

since X_u and X_v are principal directions as any tangent vectors. From which, the components of the first fundamental form are

$$E = \langle t_\alpha, t_\alpha \rangle = 1, \quad F = \langle t_\alpha, t_\beta \rangle = \cos[\phi(u,v)],$$
$$G = \langle t_\beta, t_\beta \rangle = 1, \tag{12}$$

where $\phi(u,v)$ is the angle between tangent vectors of $\alpha(u)$ and $\beta(v)$. Then, the unit normal of the translation surface can be given by

$$U(u,v) = \frac{t_\alpha \wedge t_\beta}{\sin[\phi(u,v)]}, \quad \sin[\phi(u,v)] \neq 0. \tag{13}$$

Also, the components of the second fundamental form of X are obtained by

$$\begin{cases} L = \kappa_\alpha \cos[\theta_\alpha(u,v)], \\ M = 0, \\ N = \kappa_\beta \cos[\theta_\beta(u,v)], \end{cases} \tag{14}$$

where $\theta_\alpha(u,v)$ and $\theta_\beta(u,v)$ are the angles between U and n_α, n_β, respectively.

It is worth noting that: for the degenerate curves ($\kappa = 0$), the tangent, normal and binormal are constant vectors. From this note, it is easy to prove the following theorem: