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ORIGINAL ARTICLE

A small time solutions for the KdV equation using Bubnov-Galerkin finite element method

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Abstract A Bubnov-Galerkin finite element method with quintic B-spline functions taken as element shape and weight functions is presented for the solution of the KdV equation. To demonstrate the accuracy, efficiency and reliability of the method three experiments are undertaken for both the evolution of a single solitary wave and the interaction of two solitary waves. The numerical results are compared with analytical solutions and the numerical results in the literature. It is shown that the method presented is accurate, efficient and can be used at small times when the analytical solution is not known.

1. Introduction

In this paper we consider the Korteweg-de Vries (KdV) equation in the form,

$$U_t + \varepsilon U U_x + U_{xxx} = 0 \quad a \leqslant x \leqslant b \tag{1}$$

where $U(x,t)$ is an appropriate field variable, ε and μ are positive parameters, and the subscripts t and x denote differentiation with respect to the time and the space, respectively. The KdV Eq. (1) is a one-dimensional non-linear partial differential equation (PDE) of third order, which plays a major role in the study of non-linear dispersive waves. This equation was originally derived by Korteweg-de Vries [1] to describe the behavior of one-dimensional shallow water solitary waves. Solitary waves are wave packets or pulses which propagate in non-linear dispersive media. For stable solitary wave solutions the non-linear and dispersive terms in the KdV Eq. (1) must balance, and in this case the KdV equation has traveling wave solutions called solitons. A soliton is a very special type of solitary waves which keeps its waveform after collision with other solitons.

A small time solutions using a heat balance integral (HBI) method to solve the KdV equation was obtained by Kutluay et al. [2]. In their paper, extensive comparisons with the analytical values over the defined interval are given. Bahadir [3] used the exponential finite-difference (EFD) technique to solve the KdV equation. This method has been shown to provide higher accuracy than the classical explicit finite difference and the HBI method. Ozer and Kutluay [4] used an analytical–numerical

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(AN) method, for solving the KdV equation and the obtained results are compared with that of the HBI method and the corresponding analytical solution. Irk et al. [5] used a second order spline approximation (SA) technique and made comparisons with earlier methods. Ozdes and Aksan [6] used the method of lines (MOL) for solving the KdV equation and also in [7] used a quadratic B-spline Galerkin finite element (QBGFE) method and compared these techniques with the analytical solutions and other numerical solutions that are obtained earlier using various numerical techniques.

In this paper, we present an algorithm for solving Eq. (1) by applying Bubnov Galerkin finite element method. The time integration of the resulting system is carried out using Crank–Nicholson scheme. Evolution and interaction of solitary waves with various amplitudes are undertaken.

The presence of the third spatial derivative in Eq. (1) requires that the interpolation functions and their first and second derivatives must be continuous throughout the region of solution. When using Bubnov Galerkin, the quintic B-splines interpolation functions can be used with partial differential equations containing derivatives up to order four.

The results obtained are compared with their corresponding analytical solutions and also with the various numerical methods mentioned above. To check accuracy, efficiency and reliability of the scheme presented we evaluate the invariants and error norms for the simulations undertaken.

2. Finite element scheme

A numerical solution to the KdV Eq. (1) is sought over the finite region $[a,b]$ with boundary conditions as will be prescribed. Let $a = x_0 < x_1 < \ldots < x_N = b$ be a partition of $[a,b]$ by the equally spaced knots x_i and let $\phi_i(x)$ be those quintic B-splines with knots at the points x_i, $0 < i < N$. The set of splines $\{\phi_{i-2},\phi_{i-1},\phi_i,\phi_{i+1},\phi_{i+2},\phi_{i+3}\}$ forms a basis for functions defined over the finite region $[a,b]$. We seek the approximation $U_N(x,t)$ to the solution $U(x,t)$ which uses

$$U_N(x,t) = \sum_{i=-2}^{N+2} \phi_i(x)u_i(t) \tag{2}$$

where the u_i are time dependent parameters to be determined from the boundary conditions and from conditions to be determined herein.

$$U(a,t) = U(b,t) = 0, \quad U_x(a,t) = U_x(b,t) = 0 \tag{3}$$

We identify the finite elements with the intervals $[x_i, x_{i+1}]$ with nodes at x_i and x_{i+1}. Each quintic B-splines covers six elements: consequently each element $[x_i, x_{i+1}]$ is covered by six splines $(\phi_{i-2},\phi_{i-1},\phi_i,\phi_{i+1},\phi_{i+2},\phi_{i+3})$ which are given in terms of a local coordinate system ζ given by $h\zeta = (x - x_i)$ where $h = x_{i+1} - x_i$ and $0 \leqslant \zeta \leqslant 1$. Leads to the following expressions for these splines over the element $[x_i, x_{i+1}]$ are [8,9],

$$\begin{aligned}
\phi_{i-2} &= 1 - 5\zeta + 10\zeta^2 - 10\zeta^3 + 5\zeta^4 - \zeta^5 \\
\phi_{i-1} &= 26 - 50\zeta + 20\zeta^2 + 20\zeta^3 - 20\zeta^4 + 5\zeta^5 \\
\phi_i &= 66 - 60\zeta^2 + 30\zeta^4 - 10\zeta^5 \\
\phi_{i+1} &= 26 + 50\zeta + 20\zeta^2 - 20\zeta^3 - 20\zeta^4 + 10\zeta^5 \\
\phi_{i+2} &= 1 + 5\zeta + 10\zeta^2 + 10\zeta^3 + 5\zeta^4 - 5\zeta^5 \\
\phi_{i+3} &= \zeta^5
\end{aligned} \tag{4}$$

The spline $\phi_i(x)$ and its three derivatives vanish outside the interval $[x_{i-3}, x_{i+3}]$. These spline act like ‘‘shape’’ functions for the element when we set up equations in terms of the element parameters u_i^e using Eq. (4). The variation of $U_N(x,t)$ over the element $[x_{i-3}, x_{i+3}]$ is given by

$$u^e(x,t) = \sum_{j=i-2}^{i+3} \phi_j(x)u_j(t) \tag{5}$$

The nodal value of $U_N(x, t)$ and the derivatives at the knots are given in terms of the element parameters by

$$\begin{aligned}
U_i &= u_{i-2} + 26u_{i-1} + 66u_i + 26u_{i+1} + u_{i+2}, \\
hU_i' &= 5(u_{i+2} + 10u_{i+1} - 10u_{i-1} - u_{i-2}), \\
h^2U_i'' &= 20(u_{i-2} + 2u_{i-1} - 6u_i + 2u_{i+1} + u_{i+2}), \\
h^3U_i''' &= 60(u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}), \\
h^4U_i'''' &= 120(u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}),
\end{aligned} \tag{6}$$

where the dashes denote differentiation with respect to x. An application of the Galerkin's method to Eq. (1) with weight functions $W(x)$, leads to

$$\int_a^b W(U_t + \varepsilon UU_x + \mu U_{xxx})dx = 0 \tag{7}$$

Now, we set up the relevant element matrices. For typical element $[x_i, x_{i+1}]$ we have the contribution,

$$\int_e W(u_t^e + \varepsilon u^e u_x^e + \mu u_{xxx}^e)dx$$

Replacing the weight function $W(x)$ and the unknown values $u(t)$ from (5) by B-spline shape functions (4),

$$\begin{aligned}
&\sum_{i=l-2}^{l+3}\left(\int_{x_l}^{x_{l+1}} \phi_k\phi_i dx\right)\dot{u}_i^e + \varepsilon \sum_{j=l-2}^{l+3}\sum_{i=l-2}^{l+3}\left(\int_{x_l}^{x_{l+1}} \phi_k\phi_i\phi_j' dx\right)u_i^e u_j^e \\
&\quad + \mu \sum_{i=l-2}^{l+3}\left(\int_{x_l}^{x_{l+1}} \phi_k phi_i''' dx\right)u_i^e
\end{aligned} \tag{8}$$

which in matrix form is

$$A^e\dot{u}^e + \varepsilon u^{eT}F^e u^e + \mu D^e u^e \tag{9}$$

Where

$$u^e = (u_{l-2}, u_{l-1}, u_l, u_{l+1}, u_{l+2}, u_{l+3})^T. \tag{10}$$

The element matrices are given by the integrals

$$\begin{aligned}
A_{ij}^e &= \textstyle\int_{x_l}^{x_{l+1}} \phi_i\phi_k dx, \\
F_{ijk}^e &= \textstyle\int_{x_l}^{x_{l+1}} \phi_i\phi_j'\phi_k dx, \\
D_{ij}^e &= \textstyle\int_{x_l}^{x_{l+1}} \phi_i'''\phi_j dx,
\end{aligned} \tag{11}$$

where i, j, k take only $l-2, l-1, l, l+1, l+2, l+3$ for this element $[x_l, x_{l+1}]$. The matrices A^e, D^e are therefore 6×6 and F^e is $6\times 6\times 6$. We use the associated 6×6 matrix L^e instead of F^e in our algorithm

$$L_{ij}^e = \sum_{k=l-2}^{l+3} F_{ijk}^e u_k^e, \tag{12}$$

which depends upon the parameters u_k^e. The element matrices A^e, F^e, D^e can be determined algebraically from Eq. (11), (see Appendix A), where u_k^e is given by Eq. (10). The assembly of the element Eq. (9) leads to the equation

$$Au + (\varepsilon L + \mu D)u = 0 \tag{13}$$