Journal of the Egyptian Mathematical Society (2014) **22**, 177–181

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Some integral inequalities for logarithmically convex functions

Mevlüt Tunç *

Kilis 7 Aralık University, Faculty of Science and Arts, Department of Mathematics, 79000 Kilis, Turkey

Received 26 June 2013; accepted 18 July 2013
Available online 4 September 2013

KEYWORDS

Hadamard's inequality;
Log-convex functions;
Special means

Abstract The main aim of the present note is to establish new Hadamard like integral inequalities involving log-convex function. We also prove some Hadamard-type inequalities, and applications to the special means are given.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 26A06; 26A09; 26D15; 26A20; 26B25

1. Introduction

Logarithmically convex (log-convex) functions are of interest in many areas of mathematics and science. They have been found to play an important role in the theory of special functions and mathematical statistics (see, e.g., [1–4]).

Let I be an interval of real numbers. The function $f: I \to \mathbb{R}$ is said to be *convex*on I if for all $x, y \in I$ and $t \in [0,1]$, one has the inequality:

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y). \tag{1.1}$$

A function $f:I \to (0,\infty)$ is said to be *log-convex* or *multiplicatively convex* if $\log(f)$ is convex, or equivalently, if for all x, $y \in I$ and $t \in [0,1]$, one has the inequality (see [4, p. 7]):

$$f(tx + (1-t)y) \leqslant [f(x)]^t[f(y)]^{1-t}. \tag{1.2}$$

We note that if f and g are convex functions and g is monotonic nondecreasing, then gof is convex. Moreover, since $f = exp(\log f)$, it follows that a log-convex function is convex, but the converse is not true [4, p. 7]. This fact is obvious from (1.2) as by the arithmetic-geometric mean inequality, we have

$$[f(x)]^t[f(y)]^{1-t} \leqslant tf(x) + (1-t)f(y) \tag{1.3}$$

for all x, $y \in I$ and $t \in [0,1]$.

If the above inequality (1.2) is reversed, then f is called *logarithmically concave*, or simply log-*concave*. Apparently, it would seem that log-concave (log-convex) functions would be unremarkable because they are simply related to concave (convex) functions. But they have some surprising properties. It is well known that the product of log-concave (log-convex) functions is also log-concave (log-convex). Moreover, the

* Corresponding author. Tel.: +905053317847.
E-mail address: mevluttunc@kilis.edu.tr
Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

1110-256X © 2013 Production and hosting by Elsevier B.V. on behalf of Egyptian Mathematical Society. Open access under CC BY-NC-ND license.
http://dx.doi.org/10.1016/j.joems.2013.07.004

sum of log-convex functions is also log-convex, and a convergent sequence of log-convex (log-concave) functions has a log-convex (log-concave) limit function provided that the limit is positive. However, the sum of log-concave functions is not necessarily log-concave. Due to their interesting properties, the log-convex (log-concave) functions frequently appear in many problems of classical analysis and probability theory.

The next inequality (see for example [4, p.137]) is well known in the literature as the Hermite–Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2} \tag{1.4}$$

where $f: I \to \mathbb{R}$ is a convex function on the interval I of real numbers a, $b \in I$ with $a < b$.

For some recent results related to this classic result, see the books [1–4] and the papers [5–12] where further references are given.

In [7], Dragomir and Mond proved that the following inequalities of Hermite–Hadamard type hold for log-convex functions:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leqslant \exp\left[\frac{1}{b-a}\int_a^b \ln[f(x)]dx\right] \\ &\leqslant \frac{1}{b-a}\int_a^b G(f(x), f(a+b-x))dx \\ &\leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant L(f(a), f(b)) \\ &\leqslant \frac{f(a)+f(b)}{2} \end{aligned} \tag{1.5}$$

where $G(p,q) := \sqrt{pq}$ is the geometric mean and $L(p,q) := \frac{p-q}{\ln p - \ln q}(p \neq q)$ is the logarithmic mean of the positive real numbers p, q (for $p = q$, we put $L(p,p) = p$).

In [8], Pachpatte proved that the inequalities hold for two log-convex functions:

$$\frac{4}{b-a}\int_a^b f(x)g(x)dx \leqslant [f(a)+f(b)]L(f(a), f(b)) + [g(a) + g(b)]L(g(a), g(b)) \tag{1.6}$$

In this paper, we prove another refinement of the Hermite–Hadamard Inequality for log-convex functions. Some applications for special means are also given.

Throughout this paper, we will use the following notations and conventions. Let $I \subseteq \mathbb{R} = (-\infty, +\infty)$, and a, $b \in I$ with $0 < a < b$ and

$$A(a,b) = \frac{a+b}{2}, \quad G(a,b) = \sqrt{ab}, \quad H = H(a,b) = \frac{2ab}{a+b},$$

$$L(a,b) = \frac{b-a}{\ln b - \ln a}, \quad K(a,b) = \sqrt{\frac{a^2+b^2}{2}}$$

be the arithmetic mean, geometric mean, harmonic mean, logarithmic mean, and quadratic mean, respectively.

2. Inequalities for log-convex functions

We shall start with the following refinement of the Hermite–Hadamard inequality for log-convex functions.

Theorem 1. *Let $f: I \to (0, \infty)$ be a log-convex function on I and a, $b \in I$ with $a < b$. Then, the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx \leqslant A(f(a), f(b))L(f(a), f(b)). \tag{2.1}$$

Proof. Since f is log-convex function on I, we have that

$$f(ta + (1-t)b) \leqslant [f(a)]^t [f(b)]^{1-t} \tag{2.2}$$

$$f((1-t)a + tb) \leqslant [f(a)]^{1-t} [f(b)]^t \tag{2.3}$$

for all a, $b \in I$ and $t \in [0,1]$. It is easy to observe that

$$\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx = \int_0^1 f(ta+(1-t)b)f((1-t)a+tb)dt. \tag{2.4}$$

Using the elementary inequality $G(p,q) \leqslant K(p,q)$ ($p, q \geqslant 0$ real) and making the change of variable, we get

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx \\ &\leqslant \frac{1}{2}\int_0^1 [\{f(ta+(1-t)b)\}^2 + \{f((1-t)a+tb)\}^2]dt \\ &\leqslant \frac{1}{2}\int_0^1 [\{[f(a)]^t[f(b)]^{1-t}\}^2 + \{[f(a)]^{1-t}[f(b)]^t\}^2]dt \\ &= \frac{1}{2}\left\{f^2(b)\int_0^1 \left(\frac{f(a)}{f(b)}\right)^{2t} dt + f^2(a)\int_0^1 \left(\frac{f(b)}{f(a)}\right)^{2t} dt\right\} \\ &= \frac{1}{4}\left\{f^2(b)\int_0^2 \left(\frac{f(a)}{f(b)}\right)^{u} du + f^2(a)\int_0^2 \left(\frac{f(b)}{f(a)}\right)^{u} du\right\} \\ &= \frac{1}{4}\left\{f^2(b)\left[\frac{\left(\frac{f(a)}{f(b)}\right)^u}{\log\frac{f(a)}{f(b)}}\right]_0^2 + f^2(a)\left[\frac{\left(\frac{f(b)}{f(a)}\right)^u}{\log\frac{f(b)}{f(a)}}\right]_0^2\right\} \\ &= \frac{1}{4}\left\{\frac{f^2(b)\left[\frac{f^2(a)}{f^2(b)}-1\right]}{\log f(a) - \log f(b)} + \frac{f^2(a)\left[\frac{f^2(b)}{f^2(a)}-1\right]}{\log f(b) - \log f(a)}\right\} \\ &= \frac{1}{4}\left\{\frac{f^2(a)-f^2(b)}{\log f(a) - \log f(b)} + \frac{f^2(b)-f^2(a)}{\log f(b) - \log f(a)}\right\} \\ &= \frac{1}{2}\frac{(f(a)+f(b))(f(a)-f(b))}{\log f(a) - \log f(b)} \\ &= A(f(a), f(b))L(f(a), f(b)) \end{aligned} \tag{2.5}$$

Rewriting (2.5), we get the required inequality in (2.1). The proof is complete. □

The following theorem also holds.

Theorem 2. *Let $f: I \to (0, \infty)$ be an increasing and a log-convex function on I and $a, b \in I$ with $a < b$. Then, the following inequality holds:*

$$L(f(a), f(b))f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{8(b-a)}\int_a^b f^4(x)dx + \frac{1}{8}K^2(f(a), f(b))A(f(a), f(b))L(f(a), f(b)) + 1 \tag{2.6}$$

Proof. Since f is log-convex function on I, we have that

$$f(ta + (1-t)b) \leqslant [f(a)]^t [f(b)]^{1-t} \tag{2.7}$$